Algebra

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Kapitel I

Galois theory

§ 1 Algebraic field extensions

Notations 1.1 If k, L are fields and $K \subseteq L$, L/k is called a *field extension*. The *dimension* $[L : k] := \dim_k L$ of L considered as a k-vector space, is called the *degree* of the field extension of L over k. A field extension L/k is called *finite*, if $[L : k] < \infty$. The *polynomial ring* over k is defined as

$$k[X] := \left\{ f = \sum_{i=0}^{n} a_i X^i \;\middle|\; n \geqslant 0, a_i \in k \ \forall i \in \{0, ..., n\}, a_n \neq 0 \right\} \cup \{0\}.$$

Reminder 1.2 Let L/k a field extension, $\alpha \in L$, $f \in k[X]$.

(i) $f(\alpha)$ is well defined.

(ii) $\phi_\alpha : k[X] \to L$, $f \mapsto f(\alpha)$ is a homomorphism.

(iii) $\mathrm{im}(\phi_\alpha) := k[\alpha]$ is the smallest subring of L containing k and α.

(iv) $\ker(\phi_\alpha) = \{f \in k[\alpha] \mid f(\alpha) = 0\} \lhd k[X]$ is a prime ideal.

(v) $\ker(\phi_\alpha)$ is a principle ideal.

(vi) If $f_\alpha \neq 0$ and the leading coefficient of f_α is 1, f_α is called the *minimal polynomial* of α, i.e. $f_\alpha(\alpha) = 0$ and f_α is the polynomial of smallest degree with this property. In this case, f_α is irreducible and $\ker(\phi_\alpha) = (f_\alpha)$ is a maximal ideal.

(vii) Then $L_\alpha := k[X] / \ker(\phi_\alpha) = k[X] / (f_\alpha)$ is a field.

(viii) We have $k[\alpha] = \mathrm{im}(\phi_\alpha) \cong k[X] / \ker(\phi_\alpha) = L_\alpha$, if $f_\alpha \neq 0$. Moreover $k[\alpha] = k(\alpha)$, where $k(\alpha)$ is the smallest field containing k and α. In particular, $\frac{1}{\alpha} \in k[\alpha]$.

(ix) The degree of the field extension $k[\alpha]/k$ is $[k[\alpha] : k] = \deg(f_\alpha)$.

proof. (ii) For $f, f_1, f_2 \in k[X]$, $\lambda \in k$ we have

$$(f_1 + f_2)(\alpha) = f_1(\alpha) + f_2(\alpha) \text{and} (\lambda \text{f})(\alpha) = \lambda \text{f}(\alpha)$$

(iii) Clear.

(iv) Let $f, g \in k[X]$ such that $f \cdot g, \in \ker(\phi_\alpha)$: Then

$$0 = (f \cdot g)(\alpha) = f(\alpha) \cdot g(\alpha)$$

and since L has no zero divisors, $f(\alpha) = 0$ or $g(\alpha) = 0$ and hence $f \in \ker(\phi_\alpha)$ or $g \in \ker(\phi_\alpha)$

(v) Remember that the polynomial ring is euclidean. Take $f_\alpha \in \ker(\phi_\alpha)$ of minimal degree. We will show, that $\ker(\phi_\alpha)$ is generated by f_α. Let $g \in \ker(\phi_\alpha)$ arbitrary and write

$$g = q \cdot f_\alpha + r \text{ with } q, r \in k[X], \qquad \deg(r) < \deg(f_\alpha) \text{ or } r = 0.$$

Since $r = q \cdot f_\alpha \in \ker(\phi_\alpha)$ and the choice of f_α, $\deg(r) \nless \deg(f_\alpha)$, hence $r = 0 \Rightarrow g \in (f_\alpha)$.

(vi) If $f_\alpha = g \cdot h$, either $g(\alpha) = 0$ or $h(\alpha) = 0$. As above, this implies $g \in k$ or $h \in k^\times$, i.e. f or g is irreducible. Now assume, there is and ideal $I \trianglelefteq k[X]$ satisfying $(f_\alpha) \subsetneq I \subsetneq k[K]$. Let $g \in I \backslash (f_\alpha)$, such that $(g) = I$. Such a g exists by proof of (v). Then $f_\alpha = g \cdot h$, $h \in k[X]$. This implies, that either g or h is a constant polynomial, hence a unit. In the first case, $I = k[X]$ and in the second one $I = (f_\alpha)$, which implies the claim.

(vii) We show the more general argument: If R is a ring, $\mathfrak{m} \lhd R$ a maximal ideal, then ${R}/{\mathfrak{m}}$ is a field. Let $\overline{a} \in {R}/{\mathfrak{m}}$ for some $a \in R$, $\overline{a} \neq 0$. Let $I := (\mathfrak{m}, a)$ the smallest ideal in R containing $\mathfrak{m}$ and a. Since $\overline{a} \neq 0$, hence $a \notin \mathfrak{m}$ we have $\mathfrak{m} \subsetneq I$ and since $\mathfrak{m}$ is a maximal ideal, $I = R$. Hence $1 \in I$, so we can write $1 = x + ab$ for some $x \in \mathfrak{m}$ and $b \in R$. Then we get

$$\overline{1} = \overline{x + ab} = \overline{x} + \overline{a}\overline{b} = \overline{a}\overline{b},$$

hence $\overline{a}$ is invertible in ${R}/{\mathfrak{m}}$.

(viii) Let

$$f_\alpha = \sum_{i=0}^{n} a_i X^i$$

Note, that $a_n = 1$ and $a_0 \neq 0$, since f_α is irreducible. We get

$$\begin{aligned}
\Longrightarrow \quad 0 &= f_\alpha(\alpha) = \sum_{i=0}^{n} a_i \alpha^i = a_0 + a_1\alpha + \cdots + a_n\alpha^n \\
\Longrightarrow \quad a_0 &= -\alpha \cdot \left(a_1 + a_2\alpha + \cdots + a_{n-2}\alpha^{n-2} + \alpha^{n-1}\right) \\
\Longrightarrow \quad 1 &= -\alpha \cdot \left(\frac{a_1}{a_0} + \frac{a_2}{a_0}\alpha + \cdots + \frac{a_{n-2}}{a_0}\alpha^{n-2} + \frac{1}{a_0}\alpha n - 1\right) \\
\Longrightarrow \quad \frac{1}{\alpha} &= -\frac{a_1}{a_0} - \frac{a_2}{a_0}\alpha - \cdots - \frac{a_{n-2}}{a_0}\alpha^{n-2} - \frac{1}{a_0}\alpha^{n-1}
\end{aligned}$$

Hence $\frac{1}{\alpha} \in k[X]$ and $k[X]$ is a field.

(ix) The family $\{1, \alpha, \ldots, \alpha^{n-1}\}$ forms a basis of $k[\alpha]$ as a k-vector space. □

Example 1.3 Let $k = \mathbb{Q}$, $L = \mathbb{C}$, $\alpha = 1 + i$, $\beta = \sqrt{2}$. Then the minimal polynomials of α and β are

$$f_\alpha = (X-1)^2 + 1, \quad f_\beta = X^2 - 2.$$

Proposition 1.4 (*Kronecker*) *Let k be a field, $f \in k[X]$, $\deg(f) \geqslant 1$. Then there exists a finite field extension L/k and $\alpha \in L$, such that $f(\alpha) = 0$.*

proof. W.l.o.g. we may assume, that f is irreducible, since $f = g \cdot h = 0 \Rightarrow g = 0$ or $h = 0$. Then by 1.2 $(f) = \{f \cdot g \mid g \in k[X]\}$ is a maximal ideal and $L := k/(f)$ is a field.
Clearly k is a subfield of L, since (f) does not contain any constant polynomial, i.e., if

$$\pi : k[X] \longrightarrow k[X]/(f)$$

denotes the residue map, we have $\ker(\pi) \cap k = \{0\}$, hence $\pi|_k$ is injective. Write

$$f = \sum_{i=0}^{n} a_i X^i.$$

Then we have

$$f(\pi(X)) = \sum_{i=0}^{n} a_i \pi(X)^i = \sum_{i=0}^{n} \pi(a_i)\pi(X)^i = \pi\left(\sum_{i=0}^{n} a_i X^i\right) = \pi(f) = 0,$$

hence $\alpha := \pi(X)$ is a zero of f in L. Moreover L/k is finite with degree $[L : k] = \deg(f) = n$, since $\{1, \alpha, \ldots, \alpha^{n-1}\}$ is basis of L as a k-vector space. For the independence write

$$\sum_{i=0}^{n-1} \lambda_i \alpha^i = 0, \qquad \lambda_i \in k.$$

Assume, there is $0 \leqslant j \leqslant n-1$ with $\lambda_j \neq 0$. Then the polynomial

$$g = \sum_{i=0}^{n-1} \lambda_i X^i$$

satisfies $g(\alpha) = 0$ with $\deg(g) < \deg(f)$, which is not possible by irreducibility of f. It remains to show, that L is generated by the powers of α. We have $\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0 = 0$, hence we write

$$\alpha^n = -\left(a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0\right) \in (1, \ldots, \alpha^{n-1}).$$

By induction on n, we get $\alpha^k \in (1, \ldots, \alpha^{n-1})$ for all $k \geqslant n$. □

Example 1.5 Let $k = \mathbb{Q}$, $f = X^n - a$ for some $a \in \mathbb{Q}$. For now we assume that f is irreducible (we may be able to prove this later). Then

$$L := \mathbb{Q}[X]/(f) = \mathbb{Q}[X]/(X^n - a) \cong \mathbb{Q}[\sqrt[n]{a}] = \mathbb{Q}(\sqrt[n]{a})$$

and the degree of the extension is equal to n.

Definition 1.6 Let L/k a field extension, $\alpha \in L$.

(i) α is called *algebraic over* k, if there exists $f \in \mathbb{X}[X]\backslash\{0\}$, such that $f(\alpha) = 0$.

(ii) Otherwise α is called *transcendental.*

(iii) L/k is called an *algebraic field extension*, if every $\alpha \in L$ is algebraic over k.

Proposition 1.7 *Every finite field extension L/k is algebraic.*

proof. Let $\alpha \in L$, $n := [L : k]$ the degree of L/k. Then $1, \alpha, \dots \alpha^n$ are linearly dependant over k, i.e. there exist $\lambda_0, ..., \lambda_n \in k$, $\lambda_j \neq 0$ for at least one $0 \leqslant j \leqslant n$, such that

$$\sum_{i=0}^{n} \lambda_i \alpha^i = 0.$$

Hence the polynomial

$$f = \sum_{i=0}^{n} \lambda_i X^i \neq 0$$

satifies $f(\alpha) = 0$, thus α is algebraic over k. Since α was arbitrary, L/k is algebraic. $\square$

Proposition 1.8 *Let L/k a field extension, $\alpha, \beta \in L$.*

(i) If α, β are algebraic over k, then $\alpha + \beta$, $\alpha - \beta$, $\alpha \cdot \beta$ are also algebraic over k.

(ii) If $\alpha \neq 0$ is algebraic over k, then $\frac{1}{\alpha}$ is also algebraic over k.

(iii) $k_L := \left\{\alpha \in L \middle| \alpha \text{ is algebraic over } k\right\} \subseteq L$ is a subfield of L.

proof. (i) Since $\alpha \in L$ is algebraic over $k \Rightarrow k[\alpha] = k(\alpha)$ is a finite field extension of k. Since β is algebraic over $k \Rightarrow \beta$ is algebraic over $k[\alpha]$, hence $(k[\alpha])[\beta]/k[\alpha]$ is a finite field extension. Further, we have

$$k \subseteq k[a] \subseteq (k[\alpha])[\beta] = k[\alpha, \beta].$$

Thus $k[\alpha, \beta]/k$ is algebraic with Proposition 1.5. This implies the claim, as $\alpha + \beta$, $\alpha - \beta$, $\alpha \cdot \beta \in k[\alpha, \beta]$.

(ii) If $\alpha \neq 0$, $\frac{1}{\alpha}$ is algebraic over k with part (i).

(iii) Follows from (i) and (ii). $\square$

Definition + proposition 1.9 Let k be a field, $f \in k[X]$, $\deg(f) = n$.

(i) A field extension L/k is called a *splitting field of* f, if L is the smallest field in which f decomposes into linear factors.

(ii) A splitting field $L(f)$ exists.

(iii) The field extension $L(f)/k$ is algebraic over k.

(iv) For the degree we have $[L(f) : k] \leqslant n!$.

proof.

(ii) Do this by induction on n.

n=1 Clear.

n>1 Write $f = f_1 \cdots f_r$ with irreducible polynomials $f_i \in k[X]$. Then f splits if and only every f_i splits. Hence we may assume that f is irreducible

Consider $L_1 := k/(f)$. Then f has a zero in L_1; say α. Then we have $L_1 = k[\alpha]$. Now we can write $f = (X - \alpha) \cdot g$ for some $g \in k[X]$ with $\deg(g) = n - 1$. By induction hypothesis, there exists a splitting field $L(g)$ for g. Then f splits over $L(g)[\alpha]$.

(iii) Follows by part (iv) and Proposition 1.5

(iv) Do this again by induction.

n=1 Clear.

n>1 In the notation of part (ii) we have $[k[\alpha] : k] = \deg(f) = n$. By the multiplication formula for the degree and induction hypothesis we have

$$[L(f) : k] = [L(g)[\alpha] : k] = [L(g)[\alpha] : L(g)] \cdot [L(g) : k] \leqslant n \cdot (n-1)! = n!$$

Definition + proposition 1.10 Let k be a field.

(i) k is called *algebraically closed*, if every $f \in k[X]$ splits over k.

(ii) The following statements are equivalent:

  (1) k is algebraically closed

  (2) Every nonconstant polynomial $f \in k[X]$ has a zero in k.

  (3) There is no proper algebraic field extension of k.

  (4) If $f \in k[X]$ is irreducible, then $\deg(f) = 1$.

proof. '(1) $\Rightarrow$ (2)' Let $f \in k[X]$ be a non-constant polynomial of degree n. Then f splits over k, i.e. we have a presentation

$$f = \prod_{i=0}^{n} (X - \lambda_i)$$

with $\lambda_i \in k$ for $1 \leqslant i \leqslant n$. Every λ_i is a zero. Since $n \geqslant 1$, we find a zero for any nonconstant polynomial.

'(2) $\Rightarrow$ (3)' Assume L/k is algebraic, $\alpha \in L$. Let f_α be the minimal polynomial of α. By assumption, f_α has a zero in k. Since f_α is irreducible, we must have $f_\alpha = X - \alpha$, hence $\alpha \in k$, since $f \in k[X]$.

'(3) $\Rightarrow$ (4)' Let $f \in k[X]$ irreducible. Then $L := k[X]/(f)$ is an algebraic field extension. By (3), $L = k$, hence $1 = [L : k] = \deg(f)$.

'(4) $\Rightarrow$ (1)' For $f \in k[X]$ write $f = f_1 \cdots f_r$ with irreducible polynomials f_i for $1 \leqslant i \leqslant r$. With (4), $\deg(f_i) = 1$ for any i, hence f splits. □

Lemma 1.11 *Let k be a field. Then there exists an algebraic field extension k'/k, such that every $f \in k[X]$ has a zero in k'.*

proof. For every irreducible polynomial $f \in k[X]$ introduce a symbol X_f and consider

$$R := k[\{X_f | f \in k[X] \text{ irreducible}\}] \supseteq k.$$

Monomials in R look like

$$g = \lambda \cdot X_{f_1}^{n_1} X_{f_2}^{n_2} \cdots X_{f_k}^{n_k}$$

with $\lambda \in k$, $n_i \in \mathbb{N}$. Let $I \trianglelefteq R$ be the ideal generated by the $f(X_f)$, $f \in k[X]$ irreducible. The following claims prove the lemma:

Claim (a) $I \neq R$

Claim (b) There exists a maximal ideal $\mathfrak{m} \trianglelefteq R$ containing I.

Claim (c) $k' = R/\mathfrak{m}$

To finish the proof, it remains to show the claims.

(a) Assume $I = R$. Then $1 \in I$, i.e.

$$1 = \sum_{i=1}^{k} g_{f_i} f_i (X_{f_i})$$

for suitable $g_{f_i} \in R$. Let L/k be a field extension in which all f_i have a zero α_i. Define a ring homomorphism by

$$\pi : R \longrightarrow L,\, X_f \mapsto \begin{cases} \alpha_i, & f = f_i \\ 0, & \text{otherwise} \end{cases}$$

Then we obtain

$$1 = \pi(1) = \pi \left(\sum_{i=1}^{k} g_{f_i} f_i (X_{f_i}) \right) = \sum_{i=1}^{k} \pi(g_{f_i}) f_i (\pi(X_{f_i})) = \sum_{i=1}^{k} \pi(g_{f_i}) f_i (\alpha_i) = 0,$$

hence our assumption was false and we have $I \neq R$.

(b) Let $\mathcal{S}$ be the set of all proper ideals of R containing I. By claim 2, $I \in \mathcal{S}$. Let now

$$S_1 \subseteq S_2 \subseteq S_3 \subseteq \dots$$

be elements of $\mathcal{S}$. More generally let N be a totally ordered subset of $\mathcal{S}$ and

$$S := \bigcap_{J \in N} J$$

Then $S \in \mathcal{S}$, hence $\mathcal{S}$ is nonempty. By Zorn's Lemma we know that $\mathcal{S}$ contains a maximal element $\mathfrak{m} \neq R$. Then $\mathfrak{m}$ is maximal ideal of R, since an ideal $J \trianglelefteq R$ satisfying $\mathfrak{m} \subsetneq J \subsetneq R$ is contained in $\mathcal{S}$, which is a contradiction considering the choice of $\mathfrak{m}$.

(c) Clearly k' is a field extension of k. Let $f \in k[X]$ be irreducible and $\pi : R \longrightarrow k/\mathfrak{m}$ denote the residue map. Then

$$f(X_f) \in I \subseteq \mathfrak{m}$$

i.e. we have

$$\pi(X_f) = 0$$

and thus $f(\pi(X_f)) = 0$. Hence $\pi(X_f)$ is algebraic over k.

Since k' is generated by the $\pi(X_f)$, k'/k is algebraic, which finishes the proof. □

Theorem 1.12 *Let k be a field. Then there exists an algebraic field extension $\overline{k}/k$ such that $\overline{k}$ is algebraically closed. $\overline{k}$ is called the algebraic closure of k.*

proof. By Lemma 1.9 there is an algebraic field extension k'/k, such that every $f \in k[X]$ has a zero in k'. Then let

$$k_0 := k, \quad k_1 = k_0', \quad k_2 = k_1', \quad k_{i+1} = k_i' \qquad \text{for } i \geqslant 1$$

Clearly k_i is algebraic over k for all $i \in \mathbb{N}_0$ and $k_i \subseteq k_{i+1}$. Define

$$\overline{k} := \bigcup_{i \in \mathbb{N}_0} k_i$$

Then $\overline{k}/k$ is an algebraic field extension. For $f \in \overline{k}[X]$ we find $i \in \mathbb{N}_0$ with $f \in k_i[X]$, hence f has a zero in k_i. With proposition 1.8, $\overline{k}$ is algebraically closed. □

§ 2 Simple field extensions

Definition 2.1 A field extension L/k is called *simple*, if there exists some $\alpha \in L$ such that $L = k[\alpha]$.

Example 2.2 Let $f \in k[X]$ be irreducible, $L := k[X]/(f)$. Then $L = k[\alpha]$ where $\alpha = \pi(X) = \overline{X}$ and $\pi : k[X] \longrightarrow L$ denotes the residue map. Conversely, if L/k is simple and algebraic, then $L = k[\alpha]$ for some algebraic $\alpha \in L$. Let $f \in k[X]$ be the minimal polynomial of α over k, then

$$L = k[\alpha] = k(\alpha) = k[X]/(f).$$

Proposition 2.3 *Let L be a field. Then any finite subgroup G of the multiplicative group $L^\times$ is cyclic.*

proof. Let $\alpha \in G$ be an element of maximal order, $n := \operatorname{ord}(\alpha)$. Define

$$G' := \{\beta \in G : \operatorname{ord}(\beta) | n\}$$

We first show $G' = G$ and then $G' = (\alpha)$. Let $\beta \in G$, $m := \operatorname{ord}(\beta)$. Then

$$\operatorname{ord}(\alpha\beta) = \operatorname{lcm}(m, n) \leqslant n$$

by the property of n. Thus $m|n$ and $\beta \in G'$ and hence $G \subseteq G'$. Since $G' \subseteq G$ by definition, we have $G' = G$. Let now $\gamma \in G'$. We have $\gamma^n = 1$, hence γ is zero of

$$f = X^n - 1$$

f has at most n zeros, but since $|(\alpha)| = n$, we have $(\alpha) = G'$ which finishes the proof. □

Corollary 2.4 *Let k be a finite field. Then every finite field extension L/k is simple.*

proof. We have $|L| = |k|^{[L:k]}$ and thus L is also finite. With proposition 2.2 there exists some $\alpha \in L$ such that $L^\times = L\backslash\{0\} = (\alpha)$, hence $L = k[\alpha]$, which proves the claim. □

Remark 2.5 *Let L/k be a finite field extension, $f \in k[X]$ and $\alpha \in L$ a zero of f. Let $\overline{k}$ be an algebraic closure of k and $\sigma : L \longrightarrow \overline{k}$ a homomorphism of field such that $\sigma|_k = id_k$. Then $\sigma(\alpha)$ is a zero of f.*

proof. Write

$$f = \sum_{i=0}^{n} a_i X^i$$

with coefficients $a_i \in k$, hence we have $\sigma(a_i) = a_i$ for $0 \leqslant i \leqslant n$. We obtain

$$f(\sigma(\alpha)) = \sum_{i=0}^{n} a_i (\sigma(\alpha))^i = \sum_{i=0}^{n} \sigma(a_i) (\sigma(\alpha))^i = \sigma\left(\sum_{i=0}^{n} a_i \alpha^i\right) = \sigma(f(\alpha)) = \sigma(0) = 0,$$

which finishes the proof. □

Theorem 2.6 *Let L/k be a finite field extension of degree $n := [L : k]$ and $\overline{k}$ an algebraic closure of k. If there exist n different field homomorphisms $\sigma_1, \ldots \sigma_n : k \longrightarrow L$ such that $\sigma_i|_k = id_k$, then L/k is simple.*

proof. Let $L = k[\alpha_1, ..., \alpha_r]$ for some $r \geqslant 1$ and $\alpha_i \in L$. Prove the statement by induction on r.
r=1 $L = k[\alpha_1]$, hence L is simple.
r>1 Let now $L' = k[\alpha_1, \ldots \alpha_{r-1}]$. By hypothesis, L'/k is simple, say $L = k[\beta]$. Then we have

$$L = k[\alpha_1, \ldots \alpha_r] = L'[\alpha_r] = k[\alpha, \beta]$$

with $\alpha := \alpha_r$. For $\lambda \in k$ consider

$$\gamma := \gamma_\lambda = \alpha + \lambda\beta.$$

By remark 2.4 it suffices to show

$$\sigma_i(\gamma) \neq \sigma_j(\gamma) \text{ for } i \neq j.$$

Assume there are $i \neq j$ such that $\sigma_i(\gamma) = \sigma_j(\gamma)$. Then

$$\sigma_i(\alpha) + \lambda\sigma_i(\beta) = \sigma_j(\alpha) + \lambda\sigma_j(\beta),$$

so we get

$$\sigma_i(\alpha) - \sigma_j(\alpha) + \lambda\left(\sigma_i(\beta) - \sigma_j(\beta)\right) = 0.$$

Consider the polynomial

$$g := \prod_{1 \leqslant i \neq j \leqslant n} \sigma_i(\alpha) - \sigma_j(\alpha) + X \cdot \left(\sigma_i(\beta) - \sigma_j(\beta)\right).$$

By proposition 2.2 we may assume, that k is infinite. Note that g is not the zero polynomial: If $g = 0$, we find $i \neq j$ such that $\sigma_i(\alpha) = \sigma_j(\alpha)$ and $\sigma_i(\beta) = \sigma_j(\beta)$. Since α, β generate L, σ_i and σ_j must be equal on L, which is a contradiction. Therefore we find $\lambda \in k$, such that $g(\lambda) \neq 0$. Hence the minimal polynomial m_{γ_λ} of $\gamma_\lambda = \alpha + \lambda\beta$ has at least n zeroes, i.e.

$$\deg(m_{\gamma_\lambda}) \geqslant n \Rightarrow [k[\gamma_\lambda] : k] \geqslant n$$

and hence $k[\gamma_\lambda] = L$. □

Proposition 2.7 *Let $L = k[\alpha]$ be a simple, finite field extension, $\overline{k}$ an algebraic closure of k. Let $f \in k[X]$ the minimal polynomial of α. Then for every zero β of f in $\overline{k}$ there exists a unique homomorphism of fields $\sigma : L \longrightarrow \overline{k}$ such that $\sigma(\alpha) = \beta$.*

proof. The uniqueness is clear. It remains to show the existence. Define

$$\phi_\beta : k[X] \longrightarrow \overline{k}, \qquad g \mapsto g(\beta).$$

We have $f(\beta) = 0$, thus $(f) \subseteq ker(\phi_\beta)$ and hence ϕ_β factors to a homomorphism

$$\overline{\phi_\beta} : L \cong k[X]/(f) \longrightarrow \overline{k}$$

such that $\phi_\beta = \overline{\phi_\beta} \circ \pi$ where $\pi : k[X] \longrightarrow k[X]/(f)$ denotes the residue map. Let

$$\tau : L \longrightarrow k[X]/(f)$$

be an isomorphism. Then

$$\sigma := \overline{\phi_\beta} \circ \tau : L \longrightarrow \overline{k}$$

satisfies

$$\sigma(\alpha) = \left(\overline{\phi_\beta} \circ \tau\right)(\alpha) = \overline{\phi_\beta}\left(\tau(\alpha)\right) = \overline{\phi_\beta}(\overline{X}) = \overline{\phi_\beta}\left(\pi(X)\right) = \phi_\beta(X) = \beta,$$

thus the claim. □

Corollary 2.8 *Let $f \in k[X]$ be a nonconstant polynomial. Then the splitting field of f over k is unique, i.e. any two splitting fields L, L' of f over k are isomorphic.*

proof. Let $L = k[\alpha_1, \ldots \alpha_n]$, $L' = k[\beta_1, \ldots \beta_m]$.
Assume that f is irreducible. W.l.o.g. we have $f(\alpha_1) = f(\beta_1) = 0$. By Proposition 2.6 we find field homomorphisms

$$\sigma_1 : k[\alpha_1] \longrightarrow k[\beta_2] \text{ such that } \sigma_1|_k = \mathrm{id_k} \text{ and } \alpha_1 \mapsto \beta_1$$

$$\tau_1 : k[\beta_1] \longrightarrow k[\alpha_1] \text{ such that } \tau_1|_k = \mathrm{id_k} \text{ and } \beta_1 \mapsto \alpha_1$$

Hence, since $\sigma_1 \circ \tau_1 = \mathrm{id}_{\mathrm{k}[\beta_1]}$ and $\tau_1 \circ \sigma_1 = \mathrm{id}_{\mathrm{k}[\alpha_1]}$, σ_1 and τ_1 are isomorphisms, i.e $k[\alpha_1] \cong k[\beta_1]$. By induction on n the corollary follows. □

Definition + proposition 2.9 Let L/k, L'/k be field extension.

(i) We define

$$\mathrm{Hom}_k(L, L') := \left\{\sigma : L \longrightarrow L' \text{ field homomorphism s.t. } \sigma|_k = \mathrm{id}_k\right\}$$

$$\mathrm{Aut}_k(L) := \{\sigma : L \longrightarrow L \text{ field automorphism s.t. } \sigma|_k = \mathrm{id}_k\}$$

(ii) If L/k is finite, $\overline{k}$ an algebraic closure of k, then

$$|\mathrm{Hom}_k(L, L')| \leqslant [L : k].$$

proof. Assume first $L = k[\alpha]$ for some algebraic $\alpha \in L$. Let f be the minimal polynomial of α over k, i.e. $f \in k[X]$, $\deg(f) = [L : k]$. By 2.4 and 2.6, the elements oh $\mathrm{Hom_k(L, \overline{k})}$ correspond bijectively to the zeroes of f. Then we get

$$|\mathrm{Hom}_k(L, \overline{k})| = |\{\text{zeroes of f in } \overline{k}\}| \leqslant \deg(f) = [L : k].$$

Now consider the general case. Let $L = k[\alpha_1, \ldots \alpha_n]$ and $L' = k[\alpha_1, \ldots \alpha_{n-1}] \subseteq L = L'[\alpha_n]$. By induction on n we have $|\mathrm{Hom}_k(L', \overline{k}) \leqslant [L' : k]$. Let now

$$f = \sum_{i=0}^{d} a_i X^i \in L'[X]$$

with coefficients $a_i \in L'$ be the minimal polynomial of α_n over L'. Let $\sigma \in \mathrm{Hom}_k(L, \overline{k})$ and $\sigma' = \sigma|_{L'} \in \mathrm{Hom}_k(L', \overline{k})$, $f^{\sigma'} := \sum_{i=0}^{d} \sigma'(a_i) X^i$. Then

$$f^{\sigma'}(\sigma(\alpha_n)) = \sum_{i=0}^{d} \sigma'(a_i) (\sigma(\alpha_n))^i = \sum_{i=0}^{d} \sigma(a_i) (\sigma(\alpha_n))^i = \sigma\left(\sum_{i=0}^{d} a_i \alpha_n^i\right) = 0.$$

Thus

$$|\{\mathrm{Hom}_{L'}(L,\overline{k})\}| = |\{\sigma \in \mathrm{Hom}_k(L,\overline{k})\big|\sigma|_{L'} = \mathrm{id}_{L'}\}| \leqslant \deg(f^{\sigma'}) = \deg(f) = [L':L]$$

So all in all we have

$$|\mathrm{Hom}_k(L,\overline{k})| \leqslant |\mathrm{Hom}_k(L',\overline{k})| \cdot [L:L'] \leqslant [L:L'] \cdot [L':k] = [L:k],$$

which is exactly the assignment. □

Definition 2.10 Let k be a field, $f = \sum_{i=0}^{d} a_i X^i \in k[X]$, $\overline{k}$ an algebraic closure of k, L/k an algebraic field extension.

(i) f is called *separable* over k, if f has $\deg(f)$ different roots in $\overline{k}$, i.e. there are no multiple roots.

(ii) $\alpha \in L$ is called *separable* over k, if the minimal polynomial of α over k is separable.

(iii) L/k is called *separable*, if any $\alpha \in L$ is separable over k.

(iv) We define the *formal derivative* of f by

$$f' := \sum_{i=1}^{d} i \cdot a_i X^{i-1}$$

We have well known properties of the derivative:

$$(f+g)' = f' + g', \qquad 1' = 0, \qquad (f \cdot g)' = f \cdot g' + f' \cdot g.$$

Proposition 2.11 *Let*

$$f = \prod_{i=1}^{n}(X - \alpha_i) \in k[X], \qquad a_i \in \overline{k} \text{ for } 1 \leqslant i \leqslant n$$

Then the following statements are equivalent:

(i) f *is separable.*

(ii) $(X - \alpha_i) \nmid f'$ *for* $1 \leqslant i \leqslant n$.

(iii) $\gcd(f, f') = 1$ *in* $k[X]$.

proof. '(i) ⇔ (ii)' We have

$$f' = \sum_{i=1}^{n} \prod_{j \neq i} (X - \alpha_j),$$

thus we get

$$(X - \alpha_i) \mid f' \Leftrightarrow (X - \alpha_i) \mid \prod_{j \neq i}(X - \alpha_j) \Leftrightarrow \alpha_i = \alpha_j \text{ for some } i \neq j.$$

'(ii) $\Rightarrow$ (iii)' Assume $(X-\alpha_i)\nmid f'$ for all $1\leqslant i\leqslant n$. Then

$$\gcd(f,f')=1 \text{ in } \overline{k}[X] \Longrightarrow \gcd(f,f')=1 \text{ in } k[X].$$

'(iii) $\Rightarrow$ (ii)' Let now $\gcd(f,f')=1$ in $k[X]$. Then we can write

$$1=af+bf',\ a,b\in k[X].$$

Since again $k[X]\subseteq\overline{k}[X]$, we can write $1=af+bf'$ for $a,b\in\overline{k}[X]$ an hence we obtain $\gcd(f,f')=1$ in $\overline{k}[X]$. This implies

$$(X-\alpha_i)\nmid f' \text{ for all } 1\leqslant i\leqslant n,$$

which was to be shown. □

Corollary 2.12 *(i) An irreducible polynomial $f\in k[X]$ is separable if and only if $f'\neq 0$.*
(ii) Any algebraic field extension in characteristic 0 is separable.

Example 2.13 Let $\mathrm{char}(k)=p>0$. Then

$$X^p-1=(X-1)^p$$

Let $k=\mathbb{F}_p(t)$ and $f=X^p-t\in\mathbb{F}_p(t)[X]$. Then $f'=0$, hence f is not separable, but f is irreducible in $\mathbb{F}_p(t)[X]$.

Definition + proposition 2.14 Let L/k be a finite field extension, $\overline{k}$ an algebraic closure of k and L.

(i) $[L:k]_s:=|\mathrm{Hom_k(L,\overline{k})}|$ is called the *degree of separability* of L/k.

(ii) If $L=k[\alpha]$ for some separable $\alpha\in L$ with minimal polynomial m_α over k, then

$$[L:k]_s=\deg(m_\alpha)=[L:k].$$

(iii) If $L=k[\alpha]$ for some $\alpha\in L$, $\mathrm{char}(k)=p>0$, then there exists $n\geqslant 0$, such that

$$[L:k]=p^n\cdot[L:k]_s$$

(iv) If $k\subseteq\mathbb{F}\subseteq L$ is an intermediate field extension, then

$$[L:k]_s=[L:\mathbb{F}]_s\cdot[\mathbb{F}:k]_s$$

proof. (i) This follows from Propoition 2.6:

$$[L:k]_s=|\mathrm{Hom}_k(L,\overline{k})|=|\{\text{ different zeroes of } f\}|=n=[L:k].$$

(iii) Write

$$f = \sum_{i=0}^{n} a_i X i.$$

If α is separable over k, we are done with part (ii). Otherwise by Corollary 2.11 we have

$$f' = \sum_{i=1}^{n} i \cdot a_i \cdot X^{i-1} \overset{!}{=} 0 \iff i \cdot a_i \equiv 0 \mod p \text{ for all } 0 \leqslant i \leqslant n$$

Thus we can write $f = g(X^p)$ for some $g \in k[X]$. Continue this way until we can write $f = g(X^{p^n})$ for some $n \in \mathbb{N}_0$ and separable g. Then

$$[k[\alpha] : k]_s = |\{ \text{ zeroes of } g \text{ in } \overline{k}\}| = \deg(g)$$

and thus we obtain

$$[k[\alpha] : k] = \deg(f) = \deg(g) \cdot p^n = p^n \cdot [k[\alpha] : k]_s.$$

(iv) Consider first the simple case $L = k(\alpha)$. Let

$$f = \sum_{i=0}^{n} a_i X^i \in \mathbb{F}[X]$$

be the minimal polynomial of α over $\mathbb{F}$. Let $\tau \in \mathrm{Hom}_k(\mathbb{F}, \overline{k})$ and let

$$f^\tau = \sum_{i=0}^{n} \tau(a_i) X^i.$$

Given $\sigma \in \mathrm{Hom}_k(L, \overline{k})$ with $\sigma|_{\mathbb{F}} = \tau$, notice that $\sigma(\alpha)$ is a zero of f^τ. Moreover by Proposition 2.6, every zero β of f^τ determines a unique σ such that $\sigma(\alpha) = \beta$. Thus we have

$$\begin{aligned} \left|\{\sigma \in \mathrm{Hom}_k(L, \overline{k}) \mid \sigma|_{\mathbb{F}} = \tau\}\right| &= \left|\{\beta \in \overline{k} \mid f^\tau(\beta) = 0\}\right| \\ &= \left|\{\beta \in \overline{k} \mid f(\beta) = 0\}\right| \overset{2.6}{=} [L : \mathbb{F}]_s. \end{aligned}$$

We conclude

$$\begin{aligned} [L : k]_s &= \left|\mathrm{Hom}_k(L, \overline{k})\right| = \left| \bigcup_{\tau \in \mathrm{Hom}_k(\mathbb{F}, \overline{k})} \left\{\sigma \in \mathrm{Hom}_k(L, \overline{k}) \mid \sigma|_{\mathbb{F}} = \tau\right\} \right| \\ &= \left| \left\{\sigma \in \mathrm{Hom}_k(L, \overline{k}) \mid \sigma|_{\mathbb{F}} = \tau\right\} \right| \cdot \left|\mathrm{Hom}_k(\mathbb{F}, \overline{k})\right| \\ &= [L : \mathbb{F}]_s \cdot [\mathbb{F} : k]_s \end{aligned}$$

For the general case we can write $L = \mathbb{F}(\alpha_1, \dots, \alpha_n)$. Define $L_i := \mathbb{F}(\alpha_1, \dots, \alpha_i)$, $L_0 := \mathbb{F}$

and $L_n = L$. Then L_i/L_{i-1} is simple and by the special case above we get

$$\begin{aligned}
[L:k]_s &= [L_n : L_{n-1}]_s \cdot [L_{n-1} : k]_s \\
&\vdots \\
&= [L_n : L_{n-1}]_s \cdots [L_2 : L_1]_s \cdot [L_1 : L_0]_s \cdot [L_0 : k]_s \\
&= [L_n : L_{n-1}]_s \cdots [L_2 : L_1]_s \cdot [L_1 : \mathbb{F}]_s \cdot [\mathbb{F} : k]_s \\
&= [L_n : L_{n-1}]_s \cdots [L_2 : \mathbb{F}]_s \cdot [\mathbb{F} : k]_s \\
&\vdots \\
&= [L_n : \mathbb{F}]_s \cdot [\mathbb{F} : k]_s \\
&= [L : \mathbb{F}]_s \cdot [\mathbb{F} : k]_s,
\end{aligned}$$

which implies the claim. □

Proposition 2.15 *A finite field extension L/k is separable if and only if $[L : k] = [L : k]_s$.*

proof. '⇒' Let $L = k[\alpha_1, \ldots \alpha_n]$. Prove this by induction on n.

n=1 This is proposition 12.2(ii)

n>1 Let $L' = k[\alpha_1, \ldots \alpha_{n-1}]$. Then by induction hypothesis $[L' : k]_s = [L' : k]$. Moreover $[L : L']_s = [L : L']$, since L/L' is simple by $L = L'[\alpha_n]$. By proposition 12.2 (iv) we get

$$[L : k]_s = [L : L']_s \cdot [L' : k]_s = [L : L'] \cdot [L'.k] = [L : k].$$

'⇐' Let $\alpha \in L$ and $f = m_\alpha \in k[X]$ its minimal polynomial. If $\mathrm{char}(k) = 0$, f is separable, so α is separable by corollary 2.11. Let now $\mathrm{char}(k) = p > 0$. By proposition 12.2 there exists $n \geqslant 0$ such that

$$[k[\alpha] : k] = p^n \cdot [k[\alpha] : k]_s$$

We find

$$[L : k] = [L : k[\alpha]] \cdot [k[\alpha] : k] \geqslant [L : k[\alpha]]_s \cdot p^n \, [k[\alpha] : k]_s = p^n \, [L : k]_s = p^n \, [L : k],$$

Hence we must have $n = 0$, i.e. $[k[\alpha] : k] = [k[\alpha] : k]_s$. Thus α is separable over k. □

§ 3 Galois extensions

Definition 3.1 A field extension L/k is called *normal*, if there is a subset $\mathcal{F} \subseteq k[X]$ such that L is the smallest field which any $f \in \mathcal{F}$ splits over.

Remark 3.2 *Let L/k be a normal field extension, $\overline{k}$ an algebraic closure of k. Then*

$$\mathrm{Hom}_k(L, \overline{k}) = \mathrm{Aut}_k(L).$$

proof. '$\supseteq$' Clear.

'$\subseteq$' Let L be the splitting field of $\mathcal{F}$. Let

$$f = \sum_{i=0}^{d} a_i X^i \in \mathcal{F}$$

and $\alpha \in L$ such that $f(\alpha) = 0$. Let $\sigma \in \mathrm{Hom_k(L, \overline{k})}$. Then

$$f(\sigma(\alpha)) = \sum_{i=0}^{d} a_i \sigma(\alpha)^i = \sum_{i=0}^{d} \sigma(a_i)\sigma(\alpha)^i = \sigma\left(\sum_{i=0}^{d} a_i \alpha^i\right) = \sigma(f(\alpha)) = 0,$$

hence $\sigma(\alpha)$ is zero of f. Since f splits over L, i.e. all zeroes of f are in L, we have $\sigma(\alpha) \in L$. Moreover L is generated over k by the zeroes of $f \in \mathcal{F}$, thus $\sigma(L) \subseteq L$ and hence we get $\sigma \in \mathrm{Hom}_k(L, L)$.

It remains to show bijectivity. σ is clearly injective. For the surjectivity consider that σ permutes all the zeroes of any $f \in \mathcal{F}$. Finally $\sigma \in \mathrm{Aut}_k(L)$. $\square$

Definition 3.3 An algebraic field extension L/k is called *Galois extension* or *Galois*, if it is normal and separable. In this case, the *Galois group* of L/k is defined as

$$\mathrm{Gal}(L, k) := \mathrm{Aut}_k(L).$$

Proposition 3.4 *A finite field extension L/k is Galois if and only if $|\mathrm{Aut}_k(L)| = [L : k]$.*

proof. '$\Rightarrow$' We have

$$|\mathrm{Aut}_k(L)| = |\mathrm{Hom}_k(L, \overline{k})| = [L : k]_s = [L : k]$$

'$\Leftarrow$' We have to show that L/k is separable and normal. First we see

$$[L : k] = |\mathrm{Aut_k(L)}| \leqslant |\mathrm{Hom}_k(L, \overline{k})| = [L : k]_s \leqslant [L : k]$$

Hence we have equality on each inequality, i.e. $[L : k] = [L : k]_s$ and L/k is separable.

By Theorem 2.5 we know that L/k is simple, say $L = k[\alpha]$ for some $\alpha \in L$.

Let $m_\alpha \in k[X]$ be the minimal polynomial of α over k. Moreover let $\beta \in \overline{k}$ be another zero of m_α. Then there exists $\sigma \in \mathrm{Hom}_k(L, \overline{k})$ such that $\sigma(\alpha) = \beta$. By the (in-)equality above we know $\mathrm{Aut_k(L)} = \mathrm{Hom}_k(L, \overline{k})$, hence $\sigma(\beta) \in L$. Since β was arbitrary, m_α, f splits over L, i.e. L is the splitting field of f over k. Thus L/k is normal and finally Galois. $\square$

Example 3.5 All quadratic field extensions are normal. Moreover, if $\mathrm{char}(k) \neq 2$, then all quadratic field extensions of k are Galois.

Remark 3.6 *Let L/k be a Galois extension and $k \subseteq K \subseteq L$ an intermediate field.*

(i) Then L/K is Galois and

$$\mathrm{Gal}(L/K) \leqslant \mathrm{Gal}(L/k)$$

(ii) If K/k is Galois, then $\mathrm{Gal}(L/K) \trianglelefteq \mathrm{Gal}(L/k)$ is a normal subgroup and

$$\mathrm{Gal}(L/k) \big/ \mathrm{Gal}(L/K) \cong \mathrm{Gal}(K/k).$$

proof. (i) Clearly L/K is normal, since L is the splitting field for the same polynomials as in L/k. Let now $\alpha \in L$. Then the minimal polynomial m_α of α over K divides the minimal polynomial m'_α of α over k, since $k \subseteq K$. Since m'_α has no multiple roots, m_α does not either and hence L/K is separable and thus Galois.

(ii) Define

$$\rho : \mathrm{Gal}(L/k) \longrightarrow \mathrm{Gal}(K/k),\ \sigma \mapsto \sigma|_K.$$

ρ is well defined since $\sigma|_K \in \mathrm{Hom}_K k(K, \overline{k}) = \mathrm{Aut_k(K)} = \mathrm{Gal}(K/k)$ as K/k is Galois:

$$[K : k] = |\mathrm{Aut}_k(K)| \leqslant |\mathrm{Hom}_k(K, \overline{k})| \leqslant [K : k].$$

Moreover ρ is surjective. For the kernel we get

$$\ker(\rho) = \{\sigma \in \mathrm{Gal}(L/k) \mid \sigma|_K = \mathrm{id}_K\} = \mathrm{Gal}(L/K)$$

and thus we obtain $\mathrm{Gal}(L/k) \big/ \mathrm{Gal}(L/K) \cong \mathrm{Gal}(K/k)$. □

Theorem 3.7 (*Main theorem of galois theory) Let L/k be a finite Galois extension and $G := \mathrm{Gal}(L/k)$. Then the subgroups $H \leqslant G$ correspond bijectively to the intermediate fields $k \subseteq K \subseteq L$. Explicitly we have inverse maps*

$$K \mapsto \mathrm{Gal}(L/K) \leqslant G$$

$$H \mapsto L^H := \{\alpha \in L \mid \sigma(\alpha) = \alpha \textit{ for all } \sigma \in H\}.$$

proof. Clearly L^H is a field for any $H \leqslant G$. We now have to show

(i) $\mathrm{Gal}(L/L^H) = H$ for any $H \leqslant G$.

(ii) $L^{\mathrm{Gal}(L/K)} = K$ for any intermediate field $k \subseteq K \subseteq L$.

Theese prove the theorem.

(i) We show both inclusion.

'$\supseteq$' Clear by definition.

'$\subseteq$' It suffices to show $|\mathrm{Gal}(L/L^H)| \leqslant |H|$. By 3.4(i) we have

$$|\mathrm{Gal}(L/L^H)| = [L : L^H].$$

By theorem 2.5 L/L^H is simple, say $L = L^H[\alpha]$. Define

$$f = \prod_{\sigma \in H} (X - \sigma(\alpha))$$

with $\deg(f) = |H|$. Further, since $\mathrm{id} \in H$, we have $f(\alpha) = 0$. Clearly $f \in L[X]$. We want to show that $f \in L^H[X]$. Therefore for $\tau \in H$ define

$$g^\tau := \sum_{i=0}^{n} \tau(a_i) X^i \text{ for } g = \sum_{i=0}^{n} a_i X^i$$

Then for f as defined above we have

$$f^\tau = \prod_{\sigma \in H} (X - \tau(\sigma(\alpha))) = \prod_{\sigma \in H} (X - \sigma(\alpha)) = f$$

hence $f \in L^H[X]$. From $f(\alpha) = 0$ we know that the minimal polynomial m_α of α over L^H divides f, thus

$$|\mathrm{Gal}(L/L^H)| = [L : L^H] = \deg(m_\alpha) \leqslant \deg(f) = |H|$$

(ii) '$\supseteq$' Clear by definition.

'$\subseteq$' Let $H := \mathrm{Gal}(L/K)$. Since $K \subseteq L^H$ it suffices to show $[L^H : K] = 1$. Since L^H/K is separable, this is equivalent to $[L^H : K]_s = 1$. Let now $\sigma \in \mathrm{Hom}_K(L^H, \overline{k})$. By 2.6 we can extend σ to

$$\tilde{\sigma} : L \longrightarrow \overline{k}$$

with $\tilde{\sigma}|_{L^H} = \sigma$. Explicitly: Let $L = L^H[\alpha]$ and $f \in L^H[X]$ its minimal polynomial. Choose a zero $\beta \in \overline{k}$ of f^σ. Then by 2.6 there exists $\tilde{\sigma} : L \longrightarrow \overline{k}$ with $\tilde{\sigma}(\alpha) = \beta$ and $\tilde{\sigma}|_{L^H} = \sigma$. We get $\tilde{\sigma} \in \mathrm{Gal}(L/K) = H$ and $\sigma = \tilde{\sigma}|_{L^H} = \mathrm{id}_K$ which finally implies $[L^H : K] = 1$. □

Remark 3.8 *An intermediate field* $k \subseteq K \subseteq L$ *is Galois over* k *if and only if* $\mathrm{Gal}(L/K) \trianglelefteq \mathrm{Gal}(L/k)$ *is a normal subgroup.*

proof. '$\Rightarrow$' If K/k is Galois, then $\mathrm{Gal}(L/K) = \ker(\rho)$ is a normal subgroup by 3.5.

'$\Leftarrow$' Conversely let $\mathrm{Gal}(L/K) =: H \trianglelefteq \mathrm{Gal}(L/k)$ be a normal subgroup. By 3.4 it suffices to show $\mathrm{Hom_k(K, \overline{k})} = \mathrm{Aut_k(K)}$. Let now $\sigma \in \mathrm{Hom_k(K, \overline{k})}$ and $\alpha \in K$. Extend σ to $\tilde{\sigma} : L \longrightarrow \overline{k}$. Then $\tilde{\sigma} \in \mathrm{Gal}(L/k)$. By the theorem it suffices to show that $\sigma(\alpha) \in L^{\mathrm{Gal}(L/K)} = K$, i.e. $\sigma(K) \subseteq K$. Let $\tau \in \mathrm{Gal}(L/L^H)$. Then, since $\mathrm{Gal}(L/K)$ is normal, we obtain

$$\tau(\sigma(\alpha)) = \tau(\tilde{\sigma}(\alpha)) = \left(\tilde{\sigma} \circ \tau'\right)(\alpha) = \tilde{\sigma}(\alpha) = \sigma(\alpha),$$

which implies the claim. □

Example 3.9 Let $k = \mathbb{Q}$, $f = X^5 - 4X + 2 \in \mathbb{Q}[X]$. Further let $L = L(f)$ be the splitting field of f over $\mathbb{Q}$. What is $\mathrm{Gal}(L/\mathbb{Q})$?.
We first want to show that f is irreducible. But this immediately follows by By Eisenstein's criterion for irreducibility with $p = 2$.
Thus L is an extension of $\mathbb{Q}/(f)$. Therefore $[L : \mathbb{Q}]$ is multiple of $[\mathbb{Q}/(f)] = 5$, hence $|\mathrm{Gal}(L/\mathbb{Q})|$ is divisible by 5. By Lagrange's theorem we know that $\mathrm{Gal}(L/\mathbb{Q})$ contains an element of order 5. Further note that f has exactly 3 zeroes in $\mathbb{R}$. With

$$\lim_{x\to\infty} f(x) = -\infty < 0, \qquad f(0) = 2 > 0, \qquad f(1) = -1 < 0, \qquad \lim_{x\to-\infty} f(x) = \infty > 0$$

we see by the intermediate value theorem that f has at least 3 zeroes. Moreover

$$f' = 5X^4 - 4 = 5 \cdot \left(X^4 - \frac{4}{5}\right) = 5 \cdot \left(X^2 - \frac{2}{\sqrt{5}}\right) \cdot \left(X^2 + \frac{2}{\sqrt{5}}\right)$$

Obviously, since the second factor has not real zeroes, the derivative of f has 2 zeroes, hence f has at most 3 zeroes. Together we obtain that f has exactly 3 zeroes. Since f splits over $\mathbb{C}$, f has two more conjugate zeroes in $\mathbb{C}$, say $\beta, \overline{\beta}$. Hence we know that the conjugation in $\mathbb{C}$ must be an element of $\mathrm{Gal}(L/\mathbb{Q})$.
To sum it up, we know: $\mathrm{Gal}(L/\mathbb{Q})$ is isomorphic to a subgroup of S_5, contains the conjugation, which corresponds to a transposition and moreover an element of order 5, i.e. a $5 - cycle$. But these two elements generate the whole group S_5. Hence we have $\mathrm{Gal}(L/\mathbb{Q}) \cong S_5$.

Proposition 3.10 (*Cyclotomic fields) Let k be a field, $n \in \mathbb{N}$, $\mathrm{char}(k) \nmid n$ and L the splitting field of the polynomial $f = X^n - 1$.*
Then L/k is Galois and $\mathrm{Gal}(L_n/k)$ is isomorphic to a subgroup of $(\mathbb{Z}/n\mathbb{Z})^\times$.

proof. We have $f' = nX^{n-1}$ and $f' = 0 \Leftrightarrow X = 0$ but $f(0) \neq 0$, hence f' and f_n are coprime. Thus f is separable. Since L is the splitting field of f by definition, L/k is normal, thus Galois. The zeroes of f form a group $\mu_n(k)$ under multiplication. By proposition 2.3 $\mu_n(k)$ is cyclic. Let ζ_n be a generator of $\mu_n(k)$. Define a map

$$\chi_n : \mathrm{Gal}(L_n/k) \longrightarrow (\mathbb{Z}/n\mathbb{Z})^\times \quad \sigma \mapsto m \text{ if } \sigma(\zeta_n) = \zeta_n^m$$

where m is relatively coprime to n. We obtain that χ_n is a homomorphism of groups since for $\sigma_1.\sigma_2 \in \mathrm{Gal}(L_n/k)$ we have $\sigma_2\sigma_1(\zeta_n) = \sigma_2\left(\zeta_n^{k_1}\right) = \left(\zeta_n^{k_1}\right)^{k_2} = \zeta_n^{k_1 k_2}$ and hence

$$\chi_n(\sigma_1\sigma_2) = k_1 \cdot k_2 = \chi_n(\sigma_1) \cdot \chi_n(\sigma_2).$$

Moreover χ_n is injective, since

$$\chi_n(\sigma) = 1 \Leftrightarrow \sigma(\zeta_n) = \zeta_n \Leftrightarrow \sigma = \mathrm{id}.$$

This proofs the proposition. Recall that $|(\mathbb{Z}/n\mathbb{Z})^\times| = \phi(n)$ Where ϕ is Euler's ϕ-function. $\square$

§ 4 Solvability of equations by radicals

Definition + remark 4.1 Let k be a field, $f \in k[X]$ separable.

(i) Let $L(f)$ be the splitting field of f over k. The *Galois group of the equation* $f = 0$ is defined by
$$\mathrm{Gal}(f) := \mathrm{Gal}(L(f)/k).$$

(ii) There exists an injective homomorphism of groups $\mathrm{Gal}(f) \longrightarrow S_n$ where $n := \deg(f)$.

(iii) If L/k is a finite, separable field extension, the $\mathrm{Aut_k(L)}$ is isomorphic to a subgroup of S_n, where $n = [L : k]$.

proof. (ii) Clear, since automorphisms permute the zeroes of f, of which we have at most n.

(iii) We know L/k is simple, say $L = k[\alpha]$ for some $\alpha \in L$. Let m_α be the minimal polynomial of α over k. Then $\deg(f) = n$. Every $\sigma \in \mathrm{Aut(L/k)}$ maps α to a zero of f and the same for every zero of f. Hence the claim follows. □

Definition 4.2 (i) A simple field extension $L = k[\alpha]$ of a field k is called an *elementary radical extension* if either

(1) α is a root of unity, i.e. a zero of the polynomial $X^n - 1$ for some $n \in \mathbb{N}$.

(2) α is a root of $X^n - \gamma$ for some $\gamma \in k, n \in \mathbb{N}$ such that $\mathrm{char}(k) \nmid n$.

(3) α is a root of $X^p - X - \gamma$ for somme $\gamma \in k$ where $p = \mathrm{char}(k)$.

In the following, we will denote $(1), (2)$ and (3) as the three *types* of elementary radical extensions.

(ii) A finite field extension L/k is called a *radical extension*, if there is a field extension L'/L and a chain of field extension
$$k = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_m = L'$$
such that L_i/L_{i-1} is an elementary radical extension for every $1 \leqslant i \leqslant m$.

Example 4.3 Let $k = \mathbb{Q}$, $f = X^3 - 3X + 1$. The zeroes of f (in $\mathbb{C}$) are
$$\alpha_1 = \zeta + \zeta^{-1} \in \mathbb{R},\ \alpha_2 = \zeta^2 + \zeta^{-2} \text{ and } \alpha_3 = \zeta^4 + \zeta^{-4}$$
where $\zeta = e^{\frac{2\pi i}{9}}$ is a primitive ninth root of unity. We show this exemplarily for α_1. We have
$$f(\alpha_1) = \left(\alpha_1^3 - 3\alpha_1 + 1\right) = \zeta^3 + 3\zeta + 3\zeta^{-1} + \zeta^{-3} - 3\zeta - 3\zeta^{-1} + 1 = \zeta^3 + \zeta{-}3 + 1 = 0$$
where we use $\zeta^{-3} = \overline{\zeta^{-3}}$ and since $z + \overline{z} = 2 \cdot \mathfrak{Re}(z)$ for any $z \in \mathbb{C}$ we have
$$\zeta^3 + \zeta^{-3} = 2 \cdot \mathfrak{Re}\left(\zeta^3\right) = 2 \cdot \mathfrak{Re}\left(e^{\frac{2\pi i}{3}}\right) = 2 \cdot \mathfrak{Re}\left(\cos\frac{2\pi}{3} + i \cdot \sin\frac{2\pi}{3}\right) = 2 \cdot \cos\frac{2\pi}{3} = 2 \cdot \left(-\frac{1}{2}\right) = -1.$$

Further we have

$$\alpha_1^2 = \zeta^2 + 2\zeta^{-2} + 2 = \alpha_2 + 2,$$

hence $\alpha_2 \in \mathbb{Q}(\alpha_1)$ and $\alpha_1 + \alpha_2 + \alpha_3 = 0$, hence $\alpha_3 \in \mathbb{Q}(\alpha_1, \alpha_2) = \mathbb{Q}(\alpha_1)$.
This means that $\mathbb{Q}(\alpha_1)$ contains all the zeroes of f, i.e. is a splitting field of f. We conclude

$$\mathbb{Q}(\alpha_1) \cong \mathbb{Q}/(f), \qquad [\mathbb{Q}(\alpha_1) : \mathbb{Q}] = 3.$$

From the f we see that $\mathbb{Q}(\alpha_1)/\mathbb{Q}$ is not an elementary radical extension, but a radical extension, since for $\mathbb{Q}(\zeta)$ we have $\mathbb{Q}(\alpha_1) \subseteq \mathbb{Q}(\zeta)$ and $\mathbb{Q}(\zeta)/\mathbb{Q}$ is an elementary radical extension.

Definition 4.4 Let k be afield, $f \in k[X]$ a separable, non-constant polynomial. We say f *is solvable by radicals*, if the splitting field $L(f)$ is a radical extension.

Remark 4.5 *Let L/k be an elementary field extension, referring to Definition 4.1 of type*

(1) $L = k[\zeta]$ *for some root of unity* ζ *(primitive for some suitable* $n \in \mathbb{N}$, *char*$(k) \nmid n$*). Then* L/k *is Galois with abelian Galois group*

$$\mathrm{Gal}(L/k) \cong (\mathbb{Z}/n\mathbb{Z})^\times .$$

(2) $L = k[\alpha]$ *where* α *is a root of* $X^n - \gamma$ *for some* $\gamma \in k, n \in \mathbb{N}, \mathrm{char}(k) \nmid n$. *If* k *contains the* n*-th roots of unity, i.e.* $\mu_n(\overline{k})$*, then* L/k *is Galois with cyclic Galois group.*

(3) $L = k[\alpha]$*, where* α *is a root of* $X^p - X - \gamma$ *for some* $\gamma \in k^\times$*. Then* L/k *is Galois with Galois group*

$$\mathrm{Gal}(L/k) \cong \mathbb{Z}/p\mathbb{Z}.$$

proof. (1) We proved this in proposition 3.9.

(2) Let $\zeta \in k$ be a primitive n-th root of unity. Then $\zeta^i \cdot \alpha$ is a zero of $X^n - \gamma$, where we assume n to be minimal sucht that $X^n - \gamma$ is irreducible. Then L contains all roots of $X^n - \gamma$, i.e. L/k is normal and thus Galois with

$$|\mathrm{Gal}(L/k)| = [L : k] = \deg(X^n - \gamma) = n$$

Since the automorphism $\sigma \in \mathrm{Gal}(L/k)$ that maps $\alpha \mapsto \zeta \cdot \alpha$ has order n, $\mathrm{Gal}(L/k)$ is cyclic.

(3) $f = X^p - X - \gamma$ has p zeroes in $L = k[\alpha]$. Since $f(\alpha) = 0$, we have

$$f(\alpha + 1) = (\alpha + 1)^p - (\alpha + 1) - \gamma = \alpha^p + 1 - \alpha - 1 - \gamma = \alpha^p - \alpha - \gamma = f(\alpha) = 0$$

Hence L is the splitting field of f and L/k is normal. Moreover $f' = -1 \neq 0$, hence L/k is separable and thus Galois with

$$|\mathrm{Gal}(L/k)| = [L : k] = \deg(f) = p$$

Further $\mathrm{Gal}(L/k) \ni \sigma : \ \alpha \mapsto \alpha + 1$ has order p, hence $\mathrm{Gal}(L/k)$ is cyclic and thus

$$\mathrm{Gal}(L/k) \cong \mathbb{Z}/p\mathbb{Z},$$

which is the claim. $\square$

Remark 4.6 *Let L/k be an elementary radical extension of type (ii), i.e. $L = k[\alpha]$, where α is the root of $f = X^n - \gamma$ for some $\gamma \in k, n \geqslant 1, \mathrm{char}(k) \nmid n$. $X^n - \gamma$ is irreducible*
Let $\mathbb{F}$ be a splitting field of $X^n - 1$ over k and $L\mathbb{F} = k(\alpha, \zeta)$ be the compositum of L and $\mathbb{F}$, i.e. the smallest subfield of $\overline{k}$ containing L and $\mathbb{F}$.

$$\begin{tikzcd}
 & \tilde{L} = L\mathbb{F} \arrow[dl, no head] \arrow[dr, no head] & \\
L = k[\alpha] \arrow[dr, no head] & & k[\zeta] = \mathbb{F} \arrow[dl, no head] \\
 & k &
\end{tikzcd}$$

$\tilde{L}$ is a splitting field of $X^n - \gamma$ over $\mathbb{F}$, hence $\tilde{L}/\mathbb{F}$ is Galois and by 4.4(ii), $\mathrm{Gal}(\tilde{L}/\mathbb{F})$ is cyclic. Moreover $\mathbb{F}/k$ is Galois and $\mathrm{Gal}(\mathbb{F}/k)$ is abelian. Hence $\tilde{L}/k$ is Galois and

$$\mathrm{Gal}(\tilde{L}/k) \big/ \mathrm{Gal}(\tilde{L}/\mathbb{F}) \cong \mathrm{Gal}(\mathbb{F}/k)$$

i.e. we have a short exact sequence

$$1 \longrightarrow \underbrace{\mathrm{Gal}(\tilde{L}/\mathbb{F})}_{\textit{cyclic}} \xrightarrow{\textit{inj.}} \mathrm{Gal}(\tilde{L}/k) \xrightarrow{\textit{surj.}} \underbrace{\mathrm{Gal}(\mathbb{F}/k)}_{\textit{abelian}} \longrightarrow 1.$$

Example 4.7 Let $k = \mathbb{Q}$, $f = X^3 - 2$. Then $L = \mathbb{Q}[\alpha]$ with $\alpha = \sqrt[3]{2}$ and $\mathbb{F} = \mathbb{Q}[\zeta]$ with $\zeta = e^{\frac{2\pi}{3}}$. Then $\tilde{L} = L(f)$ with $[\tilde{L} : \mathbb{Q}] = 6$. We obtain

$$\mathrm{Gal}(\tilde{L}/\mathbb{F}) \cong \mathbb{Z}/3\mathbb{Z}, \ \mathrm{Gal}(\mathbb{F}/k) \cong \mathbb{Z}/2\mathbb{Z}, \ \mathrm{Gal}(\tilde{L}/\mathbb{Q}) \cong S_3.$$

Definition 4.8 A group G is called *solvable*, if there exists a chain of subgroups

$$1 \ = \ G_0 \ \lhd G_1 \ \lhd \ \ldots \ \lhd \ G_n = G$$

where $G_{i-1} \lhd G_i$ is a normal subgroup and $G_i \big/ G_{i-1}$ is abelian for all $1 \leqslant i \leqslant n$.

Example 4.9 (i) Every abelian group is solvable.

(ii) S_4 is solvable by

$$1 \ \lhd \ V_4 \ \lhd \ A_4 \ \lhd \ S_4$$

where $V_4 = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\}$. For the quotients we have

$$V_4 \big/ \{1\} \cong \mathbb{Z}\big/2\mathbb{Z} \times \mathbb{Z}\big/2\mathbb{Z}, \qquad A_4 \big/ V_4 \cong \mathbb{Z}\big/3\mathbb{Z}, \qquad S_4 \big/ A_4 \cong \mathbb{Z}\big/2\mathbb{Z}.$$

(iii) S_5 is not solvable, since A_5 is simple (EAZ 6.6) but the quotient $A_5 \big/ \{1\}$ is not abelian.

(iv) If G, H are solvable groups, then the direct product $G \times H$ is solvable.

Proposition 4.10 *(i) Let G be a solvable group. Then*

(1) Every subgroup $H \leqslant G$ is solvable.

(2) Every homomorphic image of G is solvable.

(ii) Let

$$1 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 1$$

be a short exact sequence. Then G is solvable if and only if G' and G'' are solvable.

proof. (i) (1) Let G be solvable, i.e. we have a chain $1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$. Let $G' \leqslant G$ a subgroup. Then

$$1 \ \lhd \ G_1 \cap G' \ \lhd \ \ldots \ \lhd \ G_n \cap G' \ = \ G'$$

is a chain of subgroups of G' and we have $G_i \cap G' \lhd G_{i+1} \cap G'$ and moreover

$$(G_{i+1} \cap G')\big/(G_i \cap G') \cong G_i(G_{i+1} \cap G')\big/G_i \leqslant G_{i+1}\big/G_i.$$

Hence we have abelian quotients and G' is solvable.

(2) Let H be a group and $\phi : G \longrightarrow H$ be a surjective homomorphism of groups. Let

$$1 \ \lhd \ G_1 \ \lhd \ \ldots \ \lhd \ G_n \ = \ G.$$

Let $H_i := \phi(G_i)$. Then H_i is normal in H_{i+1}. It remains to show that the quotients are abelian. Consider

$$\begin{tikzcd} G_i \arrow[r] \arrow[d, "\phi"] & G_{i+1} \arrow[r, "\pi_G"] \arrow[d, "\phi"] \arrow[dr, dashed, "\tilde{\phi}"] & G_{i+1}\big/G_i \arrow[d, "\overline{\phi}"] \\ H_i \arrow[r] & H_{i+1} \arrow[r, "\pi_H"'] & H_{i+1}\big/H_i \end{tikzcd}$$

(We have $G_i \subseteq \ker(\tilde{\phi})$, since $\phi(G_i) = H_i = \ker(\pi_H)$. Hence $\tilde{\phi}$ factors to

$$\overline{\phi} : \underbrace{G_{i+1}\big/G_i}_{abelian} \underbrace{\longrightarrow}_{\Rightarrow} \underbrace{H_{i+1}\big/H_i}_{abelian!}$$

and we get $\overline{\phi}(a)\overline{\phi}(b) = \overline{\phi}(ab) = \overline{\phi}(ba) = \overline{\phi}(b)\overline{\phi}(a)$, hence the quotient is abelian and

$H = \phi(G)$ is solvable.

(ii) '$\Rightarrow$' Clear.

'$\Leftarrow$' Let

$$1 \lhd G_1 \lhd \cdots \lhd G_m = G', \qquad 1 \lhd H_{m+1} \lhd \cdots \lhd H_{m+k} = G''$$

chains of subgroups with abelian quotients. Define

$$G_i := \pi^{-1}\left(H_i\right)_{m+1 \leqslant i \leqslant m+k}, \ \pi : G \longrightarrow G''.$$

Then G_i is normal in G_{i+1} and we have

$$G_{m+0} = \pi^{-1}(\{1\}) = G' = G_m.$$

For $m+1 \leqslant i \leqslant m+k$ we have

$$G_{i+1}/G_i = \pi^{-1}\left(H_{i+1}/H_i\right) \cong H_{i+1}/H_i$$

and hence the chain

$$1 \lhd G_1 \lhd \cdots \lhd G_m = G' \lhd G_{m+1} \lhd \cdots \lhd G_{m+k} = G$$

reveals the solvability of G. □

Lemma 4.11 *A finite separable field extension L/k is a radical extension if and only if there exists a finite Galois extension L'/k, $L \subseteq L'$ such that* $\mathrm{Gal}(L'/k)$ *is solvable.*

proof. '$\Rightarrow$' Let

$$k = k_0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_n$$

a chain as in definition 4.7 with $L \subseteq L_n$. we prove the statement by induction.

n=1 This is exactly remark 4.5, 4.6

n>1 By induction hypothesis L_{n-1}/k is solvable. Moreover L_n/L_{n-1} is solvable, too. This is equivalent to the fact, that L_{n-1} is contained in a Galois extension $\tilde{L}_{n-1}/k$ such that $\mathrm{Gal}(\tilde{L}/k)$ is solvable and L_n is contained in a Galois extension $\tilde{L}/L_{n-1}$ such that $\mathrm{Gal}(\tilde{L}/L_{n-1})$ is solvable. We have a diagramm

$$\begin{array}{ccccccc} & & \tilde{L}_{n-1} & \subseteq & \tilde{L}L_{n-1} & := & \mathbb{M} \\ & & \cup\!| & & & & \cup\!| \\ k & \subseteq & L_{n-1} & \subseteq & L_n & \subseteq & \tilde{L} \end{array}$$

We obtain, that $\mathbb{M}$ is Galois over L_{n-1}, since $\tilde{L}, \tilde{L}_{n-1}$ are Galois over L_{n-1}, hence by

$$\iota : \mathrm{Gal}(\mathbb{M}/\tilde{L}_{n-1}) \longrightarrow \mathrm{Gal}(\tilde{L}/L_{n-1}), \ \sigma \mapsto \sigma|_{\tilde{L}}$$

an injective homomorphism of groups is given, hence

$$\mathrm{Gal}(\mathbb{M}/\tilde{L}_{n-1}) \leqslant \mathrm{Gal}(\tilde{L}/L_{n-1})$$

is solvable as a subgroup of a solvable group.
Let now $\tilde{\mathbb{M}}/\mathbb{M}$ be a minimal extension, such that $\tilde{\mathbb{M}}/k$ is Galois. Explicitly, $\tilde{\mathbb{M}}$ is defined as the *normal hull* of $\mathbb{M}$, i.e. the splitting field of the minimal polynomial of a primitive element of $\mathbb{M}/k$.
Now we want to show that Gal($\mathbb{M}/k$ is solvable.This finishes the proof of the sufficiency of our Lemma. Consider the short exact sequence

$$1 \longrightarrow \mathrm{Gal}(\tilde{\mathbb{M}}/\tilde{L}_{n-1}) \longrightarrow \mathrm{Gal}(\mathbb{M}/k) \longrightarrow \mathrm{Gal}(\tilde{L}_{n-1}/k) \longrightarrow 1.$$

By proposition 4.8 and our induction hypothesis it suffices to show that $\mathrm{Gal}(\tilde{\mathbb{M}}/\tilde{L}_{n-1})$ is solvable. Therefore observe that $\tilde{\mathbb{M}}$ is generated over k by the $\sigma(k)$ for $\sigma \in \mathrm{Hom_k}(\mathbb{M}, \overline{\mathrm{k}})$, where $\overline{k}$ denotes an algebraic closure of k. For any $\sigma \in \mathrm{Hom_k}(\mathbb{M}, \overline{\mathrm{k}})$, $\sigma(\mathbb{M})/\sigma(L_{n-1}) = \sigma(\mathbb{M})/\tilde{L}_{n-1}$ is Galois. Hence

$$\Phi : \mathrm{Gal}(\tilde{\mathbb{M}}/\tilde{L}_{n-1}) \longrightarrow \prod_{\sigma \in \mathrm{Hom_k}(\mathbb{M}, \overline{\mathrm{k}})} \mathrm{Gal}\left(\sigma(\mathbb{M})/\tilde{L}_{n-1}\right), \ \tau \mapsto \left(\tau|_{\sigma(\mathbb{M})}\right)_\sigma$$

is injective. Hence $\mathrm{Gal}(\tilde{\mathbb{M}}/\tilde{L}_{n-1})$ is solvable as a subgroup of a product of solvable groups.

'$\Leftarrow$' Let now $\tilde{L}/L$ finite such that $\mathrm{Gal}(\tilde{L}/k)$ is solvable. Let

$$1 \lhd G_1 \lhd \ldots \lhd G_n = G$$

be a chain of subgroups as in definition 4.7. By the main theorem we have bijectively correspond intermediate fields

$$\tilde{L} = L_n \supseteq L_{n-1} \supseteq \cdots \supseteq L_0 = k$$

where L_{i+1}/L_i is Galois and $\mathrm{Gal}(L_{i+1}/L) \cong \mathbb{Z}/p\mathbb{Z}$ for all $1 \leqslant i \leqslant n-1$. We now have to differ between three cases.

case 1 $p_i = \mathrm{char}(k)$. Then L_{i+1}/L_i is an elementary radical extension of type (iii), i.e. L/k is a radical extension.

case 2 $p_i \neq \mathrm{char}(k)$ *and* L_i contains a primitive p_i-th root of unity. Then L_{i+1}/L_i is an elementary radical extension of type (ii), i.e. L/k is a radical extension.

case 3 $p_i \neq \mathrm{char}(k)$ *and* L_i does not contain any primitive p_i-th root of unity. Then define

$$d := \prod_{p \in \mathbb{P}, p||G|} p$$

And let $\mathbb{F}$ be the splitting field of $X^d - 1$ over k. Then $\mathbb{F}/k$ is an elementary radical extension of type (i). Let $L' := \tilde{L}\mathbb{F}$ be the composite of $\tilde{L}$ and $\mathbb{F}$ in $\overline{k}$. Then $L'/\mathbb{F}$ is Galois by remark 4.5. Let $G' = \mathrm{Gal}(L'/\mathbb{F})$. Consider the map

$$\Psi : \mathrm{Gal}(L'/\mathbb{F}) \longrightarrow \mathrm{Gal}(\tilde{L}/k),\ \sigma \mapsto \sigma|_{\tilde{L}}.$$

Ψ is a well defined injective homomorphism of groups, hence $\mathrm{Gal}(L'/\mathbb{F}) \leqslant \mathrm{Gal}(\tilde{L}/k)$ is solvable as a subgroup of a solvable group. Let

$$1 \lhd\ G_1\ \lhd\ \dots\ \lhd\ G_n\ =\ G'$$

a chain of subgroups as in definition 4.7. Let further be

$$k \subseteq \mathbb{F} = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_n = L'$$

be the corresponding chain of intermediate fields, i.e L_i/L_{i-1} is Galois and $\mathrm{Gal}(L_i/L_{i-1}) \cong \mathbb{Z}/p\mathbb{Z}$ for $1 \leqslant i \leqslant n$. Hence, L_i/L_{i-1} is a radical extension of type (ii). Thus L/k is a radical extension, which finishes the proof. □

Theorem 4.12 *Let $f \in k[X]$ be a separable non-constant polynomial. Then f is solvable by radicals if and only if* $\mathrm{Gal}(f) = \mathrm{Gal}(L(f)/k)$ *is solvable.*

proof. Let f be solvable by radicals, i.e. $L(f)/k$ be a radical field extension.
$\iff$ $L(f)$ is contained in some Galois extension $\tilde{L}/k$ and $\mathrm{Gal}(\tilde{L}/k)$ is solvable.
$\iff$ In $k \subseteq L(f) \subseteq \tilde{L}$ all extensions are Galois.
$\overset{3.5}{\iff}$ $\mathrm{Gal}(L(f)/k) \cong \mathrm{Gal}(\tilde{L}/k) \big/ \mathrm{Gal}(\tilde{L}/L(f))$
$\overset{4.8}{\iff}$ $\mathrm{Gal}(L(f)/k)$ is solvable. □

Theorem 4.13 *Let G be a group, k a field. Then the subset* $\mathrm{Hom}(\mathrm{G}, \mathrm{k}^\times) \subseteq \mathrm{Maps}(\mathrm{G}, \mathrm{k})$ *is linearly independant in the k-vector space* $\mathrm{Maps}(\mathrm{G}, \mathrm{k})$.

proof. Suppose $\mathrm{Hom}(\mathrm{G}, \mathrm{k}^\times)$ is linearily dependant. Then let $n > 0$ minimal, such that there exist distinct elements $\chi_1, \dots \chi_n \in \mathrm{Hom}(\mathrm{G}, \mathrm{k}^\times)$ and $\lambda_1, \dots \lambda_n \in k^\times$ such that

$$\sum_{i=0}^{n} \lambda_i \chi_i = 0.$$

The χ_i are called *characters*. Clearly we have $n \geqslant 2$. Choose $g \in G$ such that $\chi_1(g) \neq \chi_2(g)$. For any $h \in G$ we have

$$0 = \sum_{i=0}^{n} \lambda_i \chi_i(gh) = \sum_{i=0}^{n} \underbrace{\lambda_i \chi_i(g)}_{=:\mu_i} \chi_i(h) = \sum_{i=0}^{n} \mu_i \chi_i(h).$$

Then we get

$$0 = \sum_{i=0}^{n} \mu_i \chi_i(h) = \sum_{i=0}^{n} \lambda_i \chi_i(g)\chi_i(h) \Rightarrow \sum_{i=0}^{n} \underbrace{(\mu_i - \lambda_i \chi_1(g))}_{=:\nu_i} \chi_i(h) = 0.$$

Consider

$$\nu_1 = \mu_1 - \lambda_1 \chi_1(g) = \lambda_1 \chi_1(g) - \lambda_1 \chi_1(g) = 0,$$

$$\nu_2 = \mu_2 - \lambda_2 \chi_1(g) = \lambda_2 \chi_2(g) - \lambda_2 \chi_1(g) = \underbrace{\lambda_2}_{\neq 0} \cdot \underbrace{(\chi_2(g) - \chi_1(g))}_{\neq 0} \neq 0.$$

Hence $\chi_2, \ldots \chi_n$ are linearly dependent. This is a contradiction to the minimality of n. □

Proposition 4.14 *Let L/k be a Galois extension such that $G := \mathrm{Gal}(L/k) = (\sigma)$ is cyclic of order d for some $\sigma \in G$, where $\mathrm{char}(k) \nmid d$. Let $\zeta_d \in k$ be a primitive d-th root of unity.*
Then there exsits $\alpha \in L^\times$ such that $\sigma(\alpha) = \zeta \cdot \alpha$.

proof. Let

$$f : L \longrightarrow L, \qquad f(X) = \sum_{i=0}^{d-1} \zeta^{-i} \cdot \sigma^i(X).$$

Applying Theorem 4.10 on $G = L^\times$ and $k = L$ shows $f \neq 0$. Then let $\gamma \in L$ such that $\alpha := f(\gamma) \neq 0$. Then we have

$$\begin{aligned} \sigma(\alpha) = \sigma(f(\gamma)) = \sigma\left(\sum_{i=0}^{d-1} \zeta^{-i} \cdot \sigma^i(\gamma)\right) &= \sum_{i=0}^{d-1} \zeta^{-i} \cdot \sigma^{i+1}(\gamma) = \zeta \cdot \sum_{i=0}^{d-1} \zeta^{-(i+1)} \cdot \sigma^{i+1}(\gamma) \\ = \zeta \cdot \sum_{i=1}^{d} \zeta^{-i} \cdot \sigma^i(\gamma) &= \zeta\left(\left(\sum_{i=1}^{d-1} \zeta^{-i} \cdot \sigma^i(\gamma)\right) + \gamma\right) \\ = \zeta \cdot f(\gamma) = \zeta \cdot \alpha. & \end{aligned}$$

Remark: The claim follows from Proposition 5.2 by insertig $\beta = \zeta$. □

Corollary 4.15 *Let L/k be a Galois extension, such that $G := \mathrm{Gal}(L/k) = (\sigma)$ is cyclic of order d for some $\sigma \in G$, where $\mathrm{char}(k) \nmid d$. Assume k contains a primitive d-th root of unity.*
Then L/k is an elementary radical extension of type (ii).

proof. Let $\zeta_d \in k$ be a primitive d-th root of unity and $\alpha \in L^\times$ such that $\sigma(\alpha) = \zeta \cdot \alpha$.
We have

$$\sigma^i(\alpha) = \zeta^i \cdot \alpha \qquad \text{for } 1 \leqslant i \leqslant d.$$

The minimal polynomial of α over k has at least d zeroes, namely $\alpha, \sigma(\alpha), \ldots \sigma^{d-1}(\alpha)$. Thus $L = k[\alpha]$. Moreover we have

$$\sigma(\alpha^d) = (\sigma(\alpha))^d = (\zeta \cdot \alpha)^d = \alpha^d,$$

hence

$$\alpha^d \in L^{(\sigma)} = L^{\mathrm{Gal}(L/k)} = k$$

where the last equation follows by the main theorem. Define $\gamma := \alpha^d$. Then the minimal polynomial of α over k is $X^d - \gamma \in k[X]$, which proves the claim. □

Proposition 4.16 *Let L/k be a Galois extension of degree $p = \mathrm{char}(k)$ with cyclic Galois group $\mathrm{Gal}(L/k) \cong \mathbb{Z}/p\mathbb{Z} = (\sigma)$. Then there exists $\alpha \in L^\times$ such that $\sigma(\alpha) = \alpha + 1$.*

proof. The proof follows by Proposition 5.4 by setting $\beta = -1$. □

Corollary 4.17 *Let L/k be a Galois extension of degree $p = \mathrm{char}(k)$ with cyclic Galois group $\mathrm{Gal}(L/k) \cong \mathbb{Z}/p\mathbb{Z} = (\sigma)$. Then L/k is an elementary radical extension of type (iii).*

proof. Let $\alpha \in L^\times$ such that $\sigma(\alpha) = \alpha + 1$. We have

$$\sigma^i(\alpha) = \alpha + i \qquad \text{for } 1 \leqslant i \leqslant p,$$

thus we have $L = k[\alpha]$. Moreover we have

$$\sigma(\alpha^p - \alpha) = \sigma^p(\alpha) - \sigma(\alpha) = (\alpha+1)^p - (\alpha+1) = \alpha^p + 1 - \alpha - 1 = \alpha^p - \alpha.$$

Thus again we have $\alpha^p \in k$. Define $\gamma := \alpha^p - \alpha$. Then the minimal polynomial of α over k is $X^p - X - \gamma$, which proves the claim. □

§ 5 Norm and trace

Definition + remark 5.1 Let L/k be a finite separable field extension, $[L : k] = n$. Let $\mathrm{Hom_k(L, \overline{k})} = \{\sigma_1, \dots \sigma_\mathrm{n}\}$.

(i) For $\alpha \in L$ we define the *norm* of α over k by

$$N_{L/k}(\alpha) := \prod_{i=1}^{n} \sigma_i(\alpha).$$

(ii) $N_{L/k} \in k$ for all $\alpha \in L$.

(iii) $N_{L/k} : L^\times \longrightarrow k^\times$ is a homomorphism of groups.

proof. (ii) Let $\alpha \in L$. Assume first that L/k is Galois. Then $\mathrm{Hom_k(L, \overline{k})} = \mathrm{Aut_k(L)} = \mathrm{Gal}(L/k)$. For $\tau \in \mathrm{Gal}(L/k)$ we have

$$\tau\left(N_{L/k}\right) = \tau\left(\prod_{i=1}^{n} \sigma_i(\alpha)\right) = \prod_{i=1}^{n} \underbrace{(\tau\sigma_i)}_{\in \mathrm{Gal}(L/k)}(\alpha) = N_{L/k},$$

hence $N_{L/k} \in L^{\mathrm{Gal}(L/k)} = k$. Now consider the general case. Let $\tilde{L} \supseteq L$ be the normal hull of L over k. Recall that $\tilde{L}$ is the composition of the $\sigma_i(L)$, i.e. we have

$$\tilde{L} = \prod_{i=1}^{n} \sigma_i(L).$$

Then $\tilde{L}/k$ is Galois an for $\tau \in \mathrm{Gal}(\tilde{L}/k)$ we have

$$\tau\left(N_{L/k}(\alpha)\right) = \prod_{i=1}^{n} \underbrace{(\tau\sigma_i)}_{\in \mathrm{Hom_k(L,\overline{k})}} (\alpha) = \prod_{i=1}^{n} \sigma_i(\alpha) = N_{L/k}(\alpha),$$

hence $N_{L/k}(\alpha) \in \tilde{L}^{\mathrm{Gal}(\tilde{L}/k)} = k$.

(iii) We have $N_{L/k}(\alpha) = 0 \iff \sigma_i(\alpha) = 0$ for some $1 \leqslant i \leqslant n \Leftrightarrow \alpha = 0$.
Moreover

$$\begin{aligned} N_{L/k}(\alpha \cdot \beta) &= \prod_{i=1}^{n} \sigma_i(\alpha\beta) = \prod_{i=1}^{n} \sigma_1(\alpha)\sigma_i(\beta) = \left(\prod_{i=1}^{n} \sigma_i(\alpha)\right) \cdot \left(\prod_{i=1}^{n} \sigma_i(\beta)\right) \\ &= N_{L/k}(\alpha) \cdot N_{L/k}(\beta), \end{aligned}$$

which proves the claim. □

Example 5.2 (i) Let $\alpha \in k$. Then

$$N_{L/k}(\alpha) = \prod_{i=1}^{n} \sigma_i(\alpha) = \prod_{i=1}^{n} \alpha = \alpha^n.$$

(ii) Let $k = \mathbb{R}$, $L = \mathbb{C}$. Then $\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \overline{\mathbb{R}}) = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{\mathrm{id}, \mathrm{z} \mapsto \overline{\mathrm{z}}\}$ and thus the norm ist $N_{L/k}(z) = z\overline{z} = |z|^2$.

(iii) Let $k = \mathbb{Q}$, $L = \mathbb{Q}[\sqrt{d}]$ for $d \in \mathbb{Z}$ squarefree. We have $[\mathbb{Q}[\sqrt{d}] : \mathbb{Q}] = 2$ and

$$\mathrm{Gal}(\mathbb{Q}[\sqrt{d}]/\mathbb{Q}) = \{\mathrm{id}, \sqrt{\mathrm{d}} \mapsto -\sqrt{\mathrm{d}}\} = \{\mathrm{a} + \mathrm{b}\sqrt{\mathrm{d}} \mapsto \mathrm{a} + \mathrm{b}\sqrt{\mathrm{d}}, \mathrm{a} + \mathrm{b}\sqrt{\mathrm{d}} \mapsto \mathrm{a} - \mathrm{b}\sqrt{\mathrm{d}}\}.$$

Then we have

$$N_{\mathbb{Q}[\sqrt{d}]/\mathbb{Q}}(a + b\sqrt{d}) = \left(a + b\sqrt{d}\right)\left(a - b\sqrt{d}\right) = a^2 - db^2$$

- $d < 0$: $d = -\tilde{d}$, hence $a^2 + \tilde{d}b^2 \overset{!}{=} 1 \Rightarrow$ either $a = \pm 1, b = 0$ or $a = 0, b = \pm 1, \tilde{d} = 1$.
- $d > 0$: Infinitely many solutions for $a^2 - bd^2 = 1$.

Proposition 5.3 (*Hilbert's theorem 90 - multiplicative version) LetL/k a finite Galois extension with cyclic Galois group* $\mathrm{Gal}(L/k) = (\sigma)$, $n = [L : k]$. *Let* $\beta \in L$ *with* $N_{L/k}(\beta) = 1$.
Then there exists $\alpha \in L^\times$ *such that* $\beta = \frac{\alpha}{\sigma(\alpha)}$.

proof. Define

$$f = \mathrm{id_L} + \beta\sigma + \beta\sigma(\beta)\sigma^2 + \ldots + \beta\sigma(\beta)\sigma^2(\beta)\cdots\sigma^{\mathrm{n}-2}(\beta)\sigma^{\mathrm{n}-1} = \sum_{\mathrm{j}=0}^{\mathrm{n}-1}\sigma^{\mathrm{j}}\prod_{\mathrm{i}=1}^{\mathrm{j}}\sigma^{\mathrm{i}-1}(\beta).$$

Then by Theorem 4.10 $f \neq 0$. Choose $\gamma \in L$ such that $\alpha := f(\gamma) \neq 0$. Then we have

$$\begin{aligned}
\beta\cdot\sigma(\alpha) &= \beta\cdot\sigma\left(f(\gamma)\right) = \beta\cdot\left(\sigma\left(\gamma + \beta\sigma(\gamma) + \ldots + \prod_{i=0}^{n-2}\sigma^i(\beta)\sigma^{n-1}(\gamma)\right)\right)\\
&= \beta\cdot\left(\sigma(\gamma) + \sigma(\beta)\sigma^2(\gamma) + \ldots + \prod_{i=0}^{n-2}\sigma^{i+1}(\beta)\sigma^n(\gamma)\right)\\
&= \beta\cdot\left(\sigma(\gamma) + \sigma(\beta)\sigma^2(\gamma) + \ldots + \frac{1}{\beta}N_{L/k}(\beta)\cdot\gamma\right)\\
&= \beta\cdot\left(\sigma(\gamma) + \sigma(\beta)\sigma^2(\gamma) + \ldots + \gamma\right)\\
&= \gamma + \beta\sigma(\gamma) + \beta\sigma(\beta)\sigma^2(\gamma) + \ldots + \beta\cdot\prod_{i=1}^{n-2}\sigma^i(\beta)\sigma^{n-1}(\gamma)\\
&= f(\gamma) = \alpha,
\end{aligned}$$

which is the claim. □

Definition + remark 5.4 Let L/k be a finite separable field extension, $[L:k] = n$. Let $\mathrm{Hom_k(L,\overline{k})} = \{\sigma_1,\ldots\sigma_{\mathrm{n}}\}$.

(i) For $\alpha \in L$,

$$tr_{L/k}(\alpha) := \sum_{i=0}^{n}\sigma_i(\alpha)$$

is called the *trace* of α over k.

(ii) $tr_{L/k}(\alpha) \in k$ for all $\alpha \in L$.

(iii) $tr_{L/k} : L \longrightarrow k$ is k-linear.

proof. (ii) As in proof 5.1, $tr_{L/k}(\alpha)$ is invariant under $\mathrm{Gal}(\tilde{L}/k)$.

(iii) Clear. □

Example 5.5 (i) Let $\alpha \in k$. Then

$$tr_{L/k}(\alpha) = \sum_{i=0}^{n}\sigma_i(\alpha) = \sum_{i=0}^{n}\alpha = n\cdot\alpha.$$

(ii) Let $k = \mathbb{R}$, $L = \mathbb{C}$. Then $tr_{\mathbb{C}/\mathbb{R}}(z) = z + \overline{z} = 2\cdot\mathfrak{Re}(z)$.

Proposition 5.6 (*Hilbert's theorem 90 - additive version) Let L/k be a Galois extension with cyclic Galois group* $\mathrm{Gal}(L/k) = (\sigma)$ *and* $[L:k] = char(k) = p \in \mathbb{P}$. *Then for every $\beta \in L$ with $tr_{L/k}(\beta) = 0$ there exists $\alpha \in L$ such that $\beta = \alpha - \sigma(\alpha)$.*

proof. Define

$$g = \beta \cdot \sigma + (\beta + \sigma(\beta)) \cdot \sigma^2 + \ldots + \left(\sum_{i=0}^{p-2} \sigma^i(\beta)\right) \cdot \sigma^{p-1} = \sum_{i=0}^{p-2} \left(\sum_{j=0}^{i} \sigma^j(\beta)\right) \cdot \sigma^{i+1}.$$

Let now $\gamma \in L$ such that $tr_{L/k}(\gamma) \neq 0$ (existing by 4.11). Then for

$$\alpha := \frac{1}{tr_{L/k}(\gamma)} \cdot g(\gamma)$$

we have

$$\begin{aligned}
\alpha - \sigma(\alpha) &= \frac{1}{tr_{L/k}(\gamma)} \cdot (g(\gamma) - \sigma(g(\gamma))) \\
&= \frac{1}{tr_{L/k}(\gamma)} \left(\left(\sum_{i=0}^{p-2} \left(\sum_{j=0}^{i} \sigma^j(\beta) \right) \sigma^{i+1}(\gamma) \right) - \left(\sum_{i=0}^{p-2} \left(\sum_{j=0}^{i} \sigma^{j+1}(\beta) \right) \sigma^{i+2}(\gamma) \right) \right) \\
&= \frac{1}{tr_{L/k}(\gamma)} \left(\left(\sum_{i=0}^{p-2} \left(\sum_{j=0}^{i} \sigma^j(\beta) \right) \sigma^{i+1}(\gamma) \right) - \left(\sum_{i=1}^{p-1} \left(\sum_{j=1}^{i} \sigma^{j}(\beta) \right) \sigma^{i+1}(\gamma) \right) \right) \\
&= \frac{1}{tr_{L/k}(\gamma)} \cdot \left(\sum_{i=0}^{p-1} \beta \cdot \sigma^i(\gamma) \right) \;=\; \beta,
\end{aligned}$$

and we obtain the claim. □

Proposition 5.7 *Let L/k be a finite separable extension, $\alpha \in L$. Consider the k-linear map*

$$\phi_\alpha : L \longrightarrow L, \quad x \mapsto \alpha \cdot x.$$

Then

(i) $N_{L/k}(\alpha) = \det(\phi_\alpha)$.

(ii) $tr_{L/k}(\alpha) = \operatorname{tr}(\phi_\alpha)$.

proof. Let

$$f = \sum_{i=0}^{d} a_i X^i$$

be the minimal polynomial of α over k. Then it holds

$$(f \circ \phi_\alpha)(x) = f(\phi_\alpha(x)) = \sum_{i=0}^{d} a_i \phi_\alpha^i(x) = \sum_{i=0}^{d} a_i \alpha^i \cdot x = x \cdot \sum_{i=0}^{d} a_i \alpha^i = x \cdot f(\alpha) = 0$$

For arbitrary $x \in L$, hence $f(\phi_\alpha) = 0$.

case 1.1 Assume first $L = k[\alpha]$ for some $\alpha \in k$. Then $[L : k] = \deg(f) = d$, so $\{1, \alpha, \ldots, \alpha^{d-1}\}$ is a k-basis of L. Then we have a transformation matrix of ϕ_α with respect to the basis $\{1, \alpha, \ldots, \alpha^{d-1}\}$

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & a_0 \\ 1 & 0 & & \vdots & -a_1 \\ 0 & 1 & & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & 1 & -a_{d-1} \end{pmatrix}$$

thus we have $\mathrm{tr}(\phi_\alpha) = -a_{d-1}$ and $\det(\phi_\alpha) = (-1)^d \cdot a_0$. We know that f splits over $\overline{k}$, say

$$f = \prod_{i=1}^{d}(X - \lambda_i) = \prod_{i=1}^{d}(X - \sigma_i(\alpha))$$

Then we easily see

$$\det(\phi_\alpha) = (-1)^d \cdot a_0 = (-1)^d \cdot f(0) = (-1)^d \cdot \prod_{i=1}^{d}(0 - \sigma_i(\alpha)) = \prod_{i=1}^{d}\sigma_i(\alpha) = N_{L/k}(\alpha),$$

$$\mathrm{tr}(\phi_\alpha) = -a_{d-1} = tr_{L/k}(\alpha).$$

case 1.2 For the case $\alpha \in k$, ϕ_α is represented by the diagonal matrix $\begin{pmatrix} \alpha & & 0 \\ & \ddots & \\ 0 & & \alpha \end{pmatrix} \in k^{d\times d}$.

We obtain

$$\mathrm{tr}(\phi_\alpha) = d \cdot \alpha = tr_{L/k}(\alpha) \qquad \det(\phi_\alpha) = \alpha^d = tr_{L/k}(\alpha).$$

case 2 For the general case we have $k \subseteq k(\alpha) \subseteq L$.

Claim (a) The following is true:

$$N_{L/k}(\alpha) = N_{k(\alpha])k}\left(N_{L/k(\alpha)}(\alpha)\right), \qquad tr_{L/k}(\alpha) = tr_{k(\alpha)/k}\left(tr_{L/k(\alpha)}(\alpha)\right)$$

Claim (b) The following identity holds:

$$\det(\phi_\alpha) = \left(\det\left(\phi_\alpha|_{k(\alpha)}\right)\right)^{[L:k(\alpha)]} \qquad \mathrm{tr}(\phi_\alpha) = [L : k(\alpha)] \cdot \mathrm{tr}\left(\phi_\alpha|_{k(\alpha)}\right).$$

Assuming Claim (a) and (b), we get

$$\begin{aligned} \det(\phi_\alpha) &= \left(\det\left(\phi_\alpha|_{k(\alpha)}\right)\right)^{[L:k(\alpha)]} \overset{1.1}{=} \left(N_{k(\alpha)/k}\right)^{[L:k(\alpha)]} = N_{k(\alpha)/k}\left(\alpha^{[L:k(\alpha)]}\right) \\ &\overset{1.2}{=} N_{k(\alpha)/k}\left(N_{L/k(\alpha)}(\alpha)\right) \\ &\overset{(a)}{=} N_{L/k}(\alpha) \end{aligned}$$

And analogously $\mathrm{tr}(\phi_\alpha) = tr_{L/k}(\alpha)$.

Let's now proof the claims.

(b) Let $x_1, \dots x_d$ be a basis of $k(\alpha)/$ as a k-vector space and $y_1, \dots y_m$ a basis of L as a $k(\alpha)$-vector space. Then the $x_i y_j$ for $1 \leqslant i \leqslant d$, $1 \leqslant j \leqslant m$ form a k-basis for L. Let now $D \in k^{d \times d}$ be the matrix representing $\phi_\alpha|_{k(\alpha)}$. Then we have

$$\alpha x_i y_j = \underbrace{(\alpha x_i)}_{\in k(\alpha)} y_j = (D \cdot x_i)\, y_j,$$

hence ϕ_α is represented by

$$\tilde{D} = \begin{pmatrix} A & 0 & \dots & 0 \\ 0 & A & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A \end{pmatrix}$$

(a) This is an exercise. □

Definition + remark 5.8 Let L/k be a finite field extension, $r = [L:k]_s = |\mathrm{Hom_k(L, \overline{k})}|$. Let $q = \frac{[L:k]}{[L:k]_s}$.

(i) For $\alpha \in L$ define

$$N_{L/k}(\alpha) = \det(\phi_\alpha) \qquad \mathrm{tr}_{L/k}(\alpha) = \mathrm{tr}(\phi_\alpha).$$

(ii) Let $\mathrm{Hom_k(L, \overline{k})} = \{\sigma_1, \dots, \sigma_\mathrm{r}\}$. Then

$$N_{L/k}(\alpha) = \left(\prod_{i=1}^{r} \sigma^i(\alpha)\right)^q, \qquad tr_{L/k}(\alpha) = \left(\sum_{i=1}^{r} \sigma_i(\alpha)\right) \cdot q.$$

proof. Copy the proof of 5.5. Recall that the minimal polynomial of α over k is given by

$$m_\alpha = \prod_{i=1}^{r} (X - \sigma_i(\alpha))^q,$$

where q is defined as above. □

§ 6 Normal series of groups

Definition 6.1 Let G be a group.

(i) A series

$$G = G_0 \; \rhd \; G_1 \; \rhd \; \dots \; \rhd \; G_n$$

of subgroups is called a *normal series* for G, if $G_i \lhd G_{i-1}$ is a normal subgroup in G_{i-1} and $G_i \neq G_{i-1}$ for $1 \leqslant i \leqslant n$. The groups $H_i := G_{i-1}/G_i$ are called *factors* of the series.

(ii) A normal series as above is called a *composition series* for G, if all its factors are simple groups and $G_n = \{e\}$.

Example 6.2 (i) For $G = S_4$ we have a composition series

$$G = S_4 \ \rhd\ A_4 \ \rhd\ V_4 \ \rhd\ T_4 \ \rhd\ \{e\}$$

where $T_4 = \{\mathrm{id}, \sigma\} \cong \mathbb{Z}/2\mathbb{Z}$ for some transposition $\sigma \in S_4$. We have quotients

$$S_4/A_4 = \mathbb{Z}/2\mathbb{Z}, \quad A_4/V_4 = \mathbb{Z}/3\mathbb{Z}, \quad V_4/T_4 = \mathbb{Z}/2\mathbb{Z}, \quad T_4/\{e\} = \mathbb{Z}/2\mathbb{Z}$$

(ii) $\mathbb{Z}$ has no composition series.
(iii) Every normal series is a composition series.
(iv) Every finite group has a composition series.

Remark 6.3 *If $G = G_0 \ \rhd\ G_1 \ \rhd\ \dots \ \rhd\ G_n = \{e\}$ is a normal composition series for a finite group G, then the following is clear:*

$$|G| = \prod_{i=1}^{n} |G_{i-1}/G_i|$$

Definition + remark 6.4 Let G be a group.

(i) For subgroups $H_1, H_2 \leqslant G$ let $[H_1, H_2]$ denote the subgroup of G generated by all *commutators*

$$[h_1, h_2] = h_1 h_2 h_1^{-1} h_2^{-1} \qquad \text{with } h_i \in H_i \text{ for } i \in \{1,2\}.$$

(ii) $[G, G] = G'$ is called the *derived* or *commutator subgroup* of G.
(iii) $G' \lhd G$ and $G^{\mathrm{ab}} := G/G'$ is abelian.
(iv) Let A be an abelian group and $\phi : G \longrightarrow A$ a homomorphism of groups. Let $\pi : G \longrightarrow G^{\mathrm{ab}}$ denote the residue map. Then $G' \subseteq \ker(\phi)$, thus ϕ factors to a unique homomorphism

$$\overline{\phi} : G^{\mathrm{ab}} \longrightarrow A, \qquad \text{such that } \phi = \overline{\phi} \circ \pi.$$

(v) The chain

$$G \ \rhd G' \ \rhd\ G'' = [G', G'] \ \rhd\ \dots \ \rhd\ G^{(n+1)} = [G^n, G^n]$$

is called the *derived series* of G.
(vi) G is solvable if and only if its derived series stops at $\{e\}$.

proof. (iii) For $g \in G$, $a, b \in G$ we have

$$g[ab]g^{-1} = gaba^{-1}b^{-1}g^{-1} = ga \underbrace{g^{-1}g}_{=e} b \underbrace{g^{-1}g}_{=e} a^{-1} \underbrace{g^{-1}g}_{=e} b^{-1}g^{-1} = [gag^{-1}, gbg^{-1}] \in G'.$$

Moreover

$$e = [\overline{a}, \overline{b}] = \overline{[a,b]} = \overline{aba^{-1}b^{-1}} \quad \Longleftrightarrow \quad \overline{ab} = \overline{a}\overline{b} = \overline{b}\overline{a} = \overline{ba}.$$

(iv) Let A be an abelian group, $\phi : G \longrightarrow A$ a himomorphism. For $x, y \in G$ we have

$$\phi([x,y]) = \phi(xyx^{-1}y^{-1}) = \phi(x) = \phi(y)\phi(x)^{-1}\phi(y)^{-1} = e \quad \Longrightarrow \quad G' \subseteq \ker(\phi).$$

(vi) '$\Leftarrow$' If the derived series of G stops at $\{e\}$, G has a normal series with abelian factors and is solvable.

'$\Rightarrow$' Let now $G = G_0 \rhd \ldots \rhd G_n = \{e\}$ be a normal series with abelian factors. We have to show that $G^{(n)} = \{e\}$.

Claim (a) We have $G^{(i)} \subseteq G_i$ for $0 \leqslant i \leqslant n$.

Then we see $G^{(n)} \subseteq G_n = \{e\}$ an hence the derived series of G stops at $\{e\}$. It remains to prove the claim.

(a) We have $\pi_i : G_i \longrightarrow G_i / G_{i+1}$ is a homomorphism from G to an abelian group. Then by part (iv), we have $G_i^{(1)} = G_i' \subseteq \ker(\pi_i) = G_{i+1}$.
By induction on n we have $G^{(i)} = (G^{(i-1)})' \subseteq G_i$, hence $\left(G^{(i)}\right)' \subseteq G_i$?.
Thus we get

$$G^{(i+1)} = \left(G^{(i)}\right)' \subseteq G_i' \subseteq \ker(\pi_I) = G_{i+1},$$

which finishes the proof. □

Proposition 6.5 *A finite group G is solvable if and only if the factors of its composition series are cyclic of prime order.*

proof. '$\Rightarrow$' Let

$$G \; = \; G_1 \; \rhd \; G_2 \; \rhd \; \ldots \; \rhd G_m \; = \; \{1\}$$

be a normal series of G with abelian quotients $G_i - 1 / G_i$ for $1 \leqslant i \leqslant m$. Refine it to a composition series

$$G \; = \; G_0 \; = H_{0,0} \rhd H_{0,1} \rhd \ldots \rhd H_{0,d_0} = \; G_1 \; = H_{1,0} \rhd \ldots \rhd H - 1, d_1 = G_2 \rhd \ldots \rhd G_m = \{1\}.$$

Then we have

$$H_{i,j} / H_{i,j+1} \; \cong \; {}^{H_{i,j} / G_{i+1}} \Big/ {}_{H_{i,j+1} / G_{i+1}} \; \subseteq \; {}^{G_i / G_{i+1}} \Big/ {}_{H_{i,j+1} / G_{i+1}}$$

hence $H_{i,j} / H_{i,j+1}$ is isomorphic to a subgroup of a factor group of an abelian group, thus abelian.

'$\Leftarrow$' Since the factor groups of the composition series are isomorphic to $\mathbb{Z} / p\mathbb{Z}$ for some primes p, the quotients are abelian, thus G is solvable. □

Theorem 6.6 (*Jordan - Hölder*) *Let G be a group and*

$$G = G_0 \rhd G_1 \rhd \ldots \rhd G_n = \{e\}$$

$$G = H_0 \rhd H_1 \rhd \ldots \rhd H_m = \{e\}$$

be two composition series of G. Then $n = m$ and there ist $\sigma \in S_n$ such that

$$H_i \big/ H_{i+1} \cong G_{\sigma(i)} \Big/ G_{\sigma(i)+1} \qquad \text{for } 0 \leqslant i \leqslant n-1$$

proof. We prove the statement by induction on n.

n=1 G is simple and thus $H_1 = \{e\}$.

n>1 Let $\overline{G} := G/G_1$ and $\pi : G \longrightarrow \overline{G}$ be the residue map.
Then $\overline{H}_i = \pi(H_i) \trianglelefteq \overline{G}$ is a normal subgroup. Since $\overline{G}$ is simple, hence we have $\overline{H}_i \in \{\{e\}, \overline{G}\}$. If $\overline{H}_1 = \overline{G}$, then $\overline{H}_2$ is a normal subgroup of $\overline{H}_1 = \overline{H}$, and so on. Hence we find $j \in \{1, \ldots m\}$ such that

$$\overline{H}_i = \overline{G} \text{ for } 0 \leqslant 1 \leqslant j \text{ and } \overline{H}_i = \{e\} \text{ for } j+1 \leqslant i \leqslant m.$$

Define $C_i := H_i \cap G_1 < G_1$ for $0 \leqslant i \leqslant m$.

Claim (a) If $j \leqslant m-2$, then we have a composition series for G_1:

$$G_1 = C_0 \rhd C_1 \rhd \ldots \rhd C_j \rhd C_{j+2} \rhd \ldots \rhd C_m = \{e\}.$$

If $j = m-1$, we have a composition series for G_1:

$$G_1 = C_0 \rhd C_1 \rhd \ldots \rhd C_{m-1} = \{e\}.$$

Clearly $G_1 \rhd G_2 \rhd \ldots \rhd G_n = \{e\}$ is a composition series, too. By induction hypothesis we have $n-1 = m-1$, hence $n = m$. Moreover we have for $i \neq j$

$$\left.\begin{array}{l} C_i \big/ C_{i+1} \cong G_{\sigma(i)} \Big/ G_{\sigma(i)+1} \\ C_j \big/ C_{j+2} \cong G_{\sigma(j)} \Big/ G_{\sigma(j)+1} \end{array}\right\} (*)$$

For some $\sigma : \{0, 1, \ldots, j, j+2, j+3, \ldots, n-1\} \longrightarrow \{1, \ldots, n-1\}$

Claim (b) We have

(1) $C_{j+1} = C_j$

(2) $C_i / C_{i+1} \cong H_i / H_{i+1}$ for $i \neq j$.

(3) $H_j / H_{j+1} \cong \overline{G} = G / G_1$.

By $(*)$ and Claim (a),(b) the theorem is proved.

It remains to show the Claims.

(a) C_{i+1} is a normal subgroup of C_i, $C_{i+1} = H_{i+1} \cap G_1$. Further C_{j+1} is normal in $C_j = C_{j+1}$

by Claim (b)(2) and ${C_i}/{C_{i+1}} \cong {H_i}/{H_{i+1}}$ for $i \neq j$ is simple by Claim (b)(2). Then ${C_j}/{C_{j+2}} = {C_j}/{C_{j+1}} = {H_j}/{H_{j+1}}$ is simple, too.

(b) (1) We have $H_{j+1} \subseteq G_1$, hence $H_{j+1} \cap G_1 = H_{j+1} = C_{j+1}$. $C_j = H_j \cap G_1$ is normal subgroup of H_j. Thus $H_j \rhd C_j \rhd C_{j+1} = H_{j+1}$. Since ${H_i}/{H_{i+1}}$ is simple, we must have $C_j = C_{j+1}$.

(2) **i>j** Then $C_i = H_i \cap G_1 = H_i$ since $H_i \subseteq G_1$.

i<j We have $\overline{H}_i = \overline{G} = {G}/{G_1}$. Then we have $G_1 H_i = G$ $(*)$, since:

'$\subseteq$' Clear.

'$\supseteq$' For $g \in G, \overline{g} \in \overline{G}$ its image there exists $h \in H_i$ such that

$$\overline{h} = \overline{g} \Longrightarrow \overline{h}^{-1}\overline{g} \in G_1 \Longleftarrow \overline{h}^{-1}\overline{g} = g_1 \in G_1 \Longrightarrow g = hg_1 \in H_i G_1.$$

With the isomorphism theorem we obtain

$${C_i}/{C_{i+1}} = {C_i}/{H_{i+1} \cap G_i} = {C_i}/{H_{i+1} \cap C_i} \cong {C_i H_{i+1}}/{H_{i+1}}.$$

Therefore it remains to show that $C_i H_{i+1} = H_i$.

'$\subseteq$' Since $C_i, H_{i+1} \subseteq H_i$ we also have $C_i H_{i+1} \subseteq H_i$

'$\supseteq$' Let $x \in H_i$. by $(*)$ we have $H_{i+1}G_i = G$. Then there exists $g \in G_1, h \in H_{i+1}$ such that $x = gh$, thus we have $g = xh^{-1} \in H_i H_{i+1} = H_i$, i.e. $g \in G_i \cap H_i = C_1$ and thus $x \in C_i H_{i+1}$.

(3) We have

$${H_i}/{H_{i+1}} = {H_i}/{C_{j+1}} = {H_j}/{C_j} = {H_j}/{H_j \cap G_1} = {G_1 H_j}/{G_1} \overset{(*)}{=} {G}/{G_1},$$

which finishes the proof, paragraph and chapter. $\square$

Kapitel II

Valuation theory

§ 7 Discrete valuations

Example 7.1 Let $P \in \mathbb{N}$ prime. For $x \in \mathbb{Z}\backslash\{0\}$ let

$$\nu_p(x) = \max\{k \in \mathbb{N} \mid p^k \mid x\}.$$

Then $p^{\nu_p(x)} \mid x, \quad p^{\nu_p(x)+1} \nmid x$. Example: $\nu_2(12) = 2$. Write $x = p^{\nu_p(x)} \cdot x'$ where $p \nmid x'$. For $\frac{x}{y} \in \mathbb{Q}^\times$ define

$$\nu_p\left(\frac{x}{y}\right) = \nu_p(x) - \nu_p(y).$$

This defines a map $\nu_p : \mathbb{Q} \longrightarrow \mathbb{Z}$, such that

(i) $v_p(ab) = \nu_p(a) + \nu_p(b)$ (clear)

(ii) $v_p(a+b) \geqslant \min\{\nu_p(a), \nu_p(b)\}$, since: Write $a = p^{\nu_p(a)} \cdot a', b = p^{\nu_p(b)} \cdot b'$. Let w.l.o.g $\nu_p(b) \leqslant \nu_p(a)$. Then we have

$$a + b = p^{\nu_p(a)} \cdot a' + p^{\nu_p(b)} \cdot b' = p^{\nu_p(b)} \cdot \left(b' + a' \cdot p^{\nu_p(a)-\nu_p(b)}\right).$$

Hence $p^{\nu_p(b)} \mid a + b$ and thus $\nu_p(a+b) \geqslant \nu_p(b) = \min\{\nu_p(a), \nu_p(b)\}$.

Definition 7.2 Let k be afield. A *discrete valuation* on k is a surjectove group homomorphism $\nu_k^\times \longrightarrow (\mathbb{Z}, +)$ satisfying

$$\nu(x+y) \geqslant \min\{\nu(x), \nu(y)\} \qquad \text{for all } x, y \in k^\times,\ x \neq -y.$$

Remark 7.3 *Let R be a factorial domain, $k = \mathrm{Quot(R)}$. Let further be $p \in R\backslash\{0\}$ be a prime element. Then $\nu_p : k^\times \longrightarrow \mathbb{Z}$ can be defined as in Example 7.1: Write*

$$x = e \cdot \prod_{p \in \mathbb{P}} p^{\nu_p(x)}, \qquad e \in R^\times$$

where $\mathbb{P}$ denotes set of representatives of prime elements of R. Then ν_p is a discrete valuation on k.

Example 7.4 Let k be a field, $a \in k$, $R = k[X]$ and $p_a = X - a \in k[X]$. For $f \in k[X]$ define $\nu_{p_a}(f) = n$ if f has an n-fold root in a, i.e. $f = (X-a)^n \cdot g$ for some $0 \neq g \in k[X]$. Then ν_{p_a} is a discrete valuation on $k(X) = \mathrm{Quot}(\mathrm{k}[X])$ satisfying $\nu_p|_k = 0$.

Remark 7.5 *There is no discrete valuation on $\mathbb{C}$.*

proof. Assume there exists a discrete valuation on $\mathbb{C}$, say $\nu : \mathbb{C}^\times \longrightarrow \mathbb{Z}$. Since ν is surjective, there exists $z \in \mathbb{C}^\times$ such that $\nu(z) = 1$.
Let now $y \in \mathbb{C}^\times$ such that $y^2 = z$. Then we have

$$1 = \nu(z) = \nu(y^2) = \nu(y \cdot y) = \nu(y) + \nu(y) = 2\nu(y) \quad \Longleftrightarrow \quad \nu(y) = \frac{1}{2} \notin \mathbb{Z}$$

which is a contradiction. □

Example 7.6 Let $\nu : \mathbb{Q}^\times \longrightarrow \mathbb{Z}$ be a nontrivial discrete valuation. Then there exists $a \in \mathbb{Z}$ such that $\nu(a) \neq 0$ and hence we find $p \in \mathbb{P}$: $\nu(p) \neq 0$.
If $\nu(q) = 0$ for all $q \in \mathbb{P}$, then $\nu = \nu_p$.
Assume we have $\nu(p) \neq 0 \neq \nu(q)$ for some $p \neq q \in \mathbb{P}$ and write $1 = ap + bq$ for suitable $a, b \in \mathbb{Z}$. Then

$$0 = \nu(1) = \nu(ap{+}bq) \geqslant \min\{\nu(ap), \nu(bq)\} = \min\{\underbrace{\nu(a)}_{\geqslant 0\ (*)} + \nu(p), \underbrace{\nu(b)}_{\geqslant 0\ (*)} + \nu(q)\} \geqslant \min\{\nu(p), \nu(q)\} > 0$$

Hence a contradiction, i.e. we have $\nu(p) \neq 0$ for at most one $p \in \mathbb{P}$, thus $\nu = \nu_p$.
$(*)$ obtain that we have $\nu(1) = \nu(1 \cdot 1) = \nu(1) + \nu(1) \Rightarrow \nu(1) = 0$ and by induction

$$\nu(a) = \nu(1 + (a-1)) \geqslant \min\{\nu(1), \nu(a-1)\} \geqslant 0$$

Proposition 7.7 *Let k be a field and $\nu : k^\times \longrightarrow \mathbb{Z}$ be a discrete valuation on k.*

(i) $\nu(1) = \nu(-1) = 0$.

(ii) $\mathcal{O}_\nu := \{x \in k^\times \mid \nu(x) \geqslant 0\} \cup \{0\}$ *is a ring, called the valuation ring of ν.*

(iii) $\mathfrak{m}_\nu := \{x \in k^\times \mid \nu(x) > 0\} \cup \{0\} \lhd \mathcal{O}_\nu$ *is an ideal in $\mathcal{O}_\nu$, called the valuation ideal of ν. More precisely, $\mathfrak{m}_\nu$ is the only maximal ideal in $\mathcal{O}_\nu$, i.e. $\mathcal{O}_\nu$ is a local ring.*

(iv) $\mathfrak{m}_\nu$ *is a principal ideal.*

(v) $\mathcal{O}_\nu$ *is a principal ideal domain. More precisely, any ideal $I \neq \{0\}$ in $\mathcal{O}_\nu$ is of the form $I = (t^d)$ for some $d \in \mathbb{N}$ and $t \in \mathfrak{m}_\nu$ with $\nu(t) = 1$.*

(vi) *We have $k = \mathrm{Quot}(\mathrm{R})$ and for $x \in k^\times$: $x \in \mathcal{O}_\nu$ or $\frac{1}{x} \in \mathcal{O}_\nu$.*

proof. (ii) This is strict calculating, which may be verified by the reader.

(iii) $\mathfrak{m}_\nu$ is an ideal, since for $x, y \in \mathfrak{m}_\nu, \alpha \in \mathcal{O}_\nu$ we have

$$\nu(x+y) \geqslant \min\{\nu(x), \nu(y)\} > 0, \qquad \nu(\alpha x) = \underbrace{\nu(\alpha)}_{\geqslant 0} + \nu(x) \geqslant \nu(x) > 0.$$

Let now $x \in \mathcal{O}_\nu$ with $\nu(x) = 0$. Then

$$\nu\left(\frac{1}{x}\right) = \nu(1) - \nu(x) = -\nu(x) = 0,$$

hence $x \in \mathcal{O}_\nu^\times$. Thus we have $\mathfrak{m}_\nu = \mathcal{O}_\nu \backslash \mathcal{O}_\nu^\times$ and the claim follows.

(iv) Let $t \in \mathfrak{m}_\nu$ such that $\nu(t) = 1$. Then for $x \in \mathfrak{m}_\nu$ let $\nu(x) = d > 0$. Then we have

$$\nu\left(x \cdot t^{-d}\right) = \nu(x) + \nu\left(\frac{1}{t^d}\right) = d + 0 - d = 0$$

Define $e := x \cdot t^{-d} \in \mathcal{O}_\nu^\times$. Then $x = e \cdot t^d$, hence $\mathfrak{m}_\nu = (t)$.

(v) Let $\{0\} \neq I \neq \mathcal{O}_\nu$ be an ideal in $\mathcal{O}_\nu$. Let $d := \min\{\nu(x) \mid x \in I \backslash \{0\}\} > 0$.

'$\supseteq$' Let $x \in I$ such that $\nu(x) = d$. By part (iv) we have $x = e \cdot t^d$ for some $e \in \mathcal{O}_\nu^\times$, hence we have $t^d \in I$; thus $(t^d) \subseteq I$.

'$\subseteq$' Let now $y \in I \backslash \{0\}$ and write $y = e \cdot t^{\nu(y)}$ for some $e \in \mathcal{O}_\nu^\times$ and $\nu(y) > d$. Then $y = t^d \cdot e \cdot t^{\nu(y)-d}$, hence $y \in (t^d)$ and thus $I \subseteq (t^d)$.

(vi) If $\nu(x) \geqslant 0$, then $x \in \mathcal{O}_\nu$. If $\nu(x) < 0$, we have

$$\nu\left(\frac{1}{x}\right) = \nu(1) - \nu(x) = -\nu(x) > 0, \qquad \text{hence } \frac{1}{x} \in \mathfrak{m}_\nu \subseteq \mathcal{O}_\nu,$$

which we wanted to show. □

Definition 7.8 An integral domain R is called a *discrete valuation ring*, if there exists a discrete valuation ν of $k = \mathrm{Quot(R)}$ such that $R = \mathcal{O}_\nu$.

Proposition 7.9 *Let R be a lokal integral domain. Then the following statements are equivalent.*

(i) R is a discrete valuation ring.

(ii) R is a principal ideal domain.

(iii) There exists $t \in R \backslash \{0\}$ such that every $x \in R \backslash \{0\}$ can uniquely be written in the form

$$x = e \cdot t^d \qquad \textit{for some } e \in R^\times,\ d \geqslant 0$$

proof. '(i) $\Rightarrow$ (ii)' This follows by 7.7.

'(ii) $\Rightarrow$ (iii)' We know that principal ideal domains are factorial. Let $t \in R$ be a generator of the maximal ideal $\mathfrak{m}$ of R. Then t is prime, since any maximal ideal is also prime. Let now $p \in R \backslash \{0\}$ a prime element. Then $p \notin R^\times$, hence $p \in \mathfrak{m}$, thus we can write $p = t \cdot x$ for some $x \in R$. Since p is prime, hence irreducible, we have $x \in R^\times \Rightarrow (p) = (t)$. Thus we have $p = t$ and we have only one prime element in R. The unique prime factorization in factorial domains gives us $x = e \cdot t^d$ for some $e \in R^\times$ and $d \geqslant 0$.

'(iii)⇒(i)' For $x = e \cdot t^d \in R\backslash\{0\}$, $e \in R^\times, d \geqslant 0$ define $\nu(x) = d$. We claim that ν is discrete valuation. We have

$$\nu(xy) = \nu\left(et^d \cdot e't^{d'}\right) = \nu\left(ee't^{d+d'}\right) = \nu\left(e''t^{d+d'}\right) = d + d'.$$

Let w.l.o.g. $d \leqslant d'$. Then

$$\nu(x+y) = \nu\left(et^d + e't^{d'}\right) = \nu\left(t^d\left(e + e't^{d'-d}\right)\right) \geqslant d = \min\{d, d'\}$$

which we extend to

$$\nu : k^\times \longrightarrow \mathbb{Z}, \qquad \nu\left(\frac{x}{y}\right) = \nu(x) - \nu(y).$$

This is well defined: For $\frac{x}{y} = \frac{x'}{y'}$ we have $xy' = x'y$ and $\nu(x'y) = \nu(x) + \nu(y') = \nu(x') + \nu(y)$, thus

$$\nu\left(\frac{x}{y}\right) = \nu(x) - \nu(y) = \nu(x') - \nu(y') = \nu\left(\frac{x'}{y'}\right).$$

Finally we have $\nu(t) = 1$, hence $\nu : k^\times \longrightarrow \mathbb{Z}$ is surjective. Thus ν is a discrete valuation on k and $R = \mathcal{O}_\nu$. □

Definition + proposition 7.10 Let R be a local ring with maximal ideal $\mathfrak{m}$.

(i) $k := R/\mathfrak{m}$ is called the *residue field* of R.

(ii) $\mathfrak{m}/\mathfrak{m}^2$ has a structure of a k-vector space.

(iii) If R is a discrete valuation ring, then $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = 1$.

proof. (ii) For $a \in R$, $x \in \mathfrak{m}$ define $\overline{a}\overline{x} = \overline{ax}$, where $\overline{a}, \overline{x}$ are the images of a, x in k.

This is well defined: Let $a' \in R$ with $\overline{a'} = \overline{a}$ and $x' \in \mathfrak{m}$ with $\overline{x'} = \overline{x}$. We have to show that

$$\overline{a'x'} = \overline{ax} \iff a'x' - ax \in \mathfrak{m}^2$$

We have $\overline{a'} = \overline{a}$, hence $a' = a + y$ for some $y \in \mathfrak{m}$. Analogously we have $\overline{x'} = \overline{x}$, hence $x' = x+$ for some $z \in \mathfrak{m}^2$. Thus we have

$$a'x' = (a+y)(b+z) = ax + az + xy + yz \equiv ax \mod \mathfrak{m}^2,$$

which finishes the proof. □

§ 8 The Gauß Lemma

Let R be a UFD (unique factorization domain), $\mathbb{P}$ a set of representatives of the primes in R with respect to *associateness*, i.e. $x \sim y \Leftrightarrow y = u \cdot x$ for some $u \in R^\times$. Every $x \in R\backslash\{0\}$ has a unique factorization

$$x = u \cdot \prod_{p \in \mathbb{P}} p^{\nu_p(x)}, \qquad \nu_p(x) \geqslant 0 \text{ for } p \in \mathbb{P},\ u \in R^\times$$

where $\nu_p : k^\times \longrightarrow \mathbb{Z}$ is a discrete valuation on $k = \mathrm{Quot}(R)$.

Definition + proposition 8.1 Let R be a factorial domain, $k = \mathrm{Quot}(R)$ and

$$f = \sum_{i=0}^{n} a_i X^i \in k[X]\backslash\{0\}, \qquad a_n \neq 0.$$

(i) For $p \in \mathbb{P}$ let $\nu_p(f) = \min\{\nu_p(a_i) \mid 0 \leqslant i \leqslant n\}$.
(ii) f is called *primitive*, if $\nu_p(f) = 0$ for all $p \in \mathbb{P}$.
(iii) If f is primitive, then $f \in R[X]$.
(iv) If $f \in R[X]$ is monic, i.e. $a_n = 1$, then f is primitive.
(v) There exists $c \in k^\times$ such that $c \cdot f$ is primitive.

proof. (iii) If f is primitive, we have $\min_{1 \leqslant i \leqslant n}\{\nu_p(a_i)\} = 0$, i.e. $\nu_p(a_i) \geqslant 0$ for all $1 \leqslant i \leqslant n$. Thus $a_i \in R$ and $f \in R[X]$.

(iv) If $a_i \in R$ we have $\nu_p(a_i) \geqslant 0$ for all $1 \leqslant i \leqslant n$. Moreover $\nu_p(a_n) = \nu_p(1) = 0$, hence $\nu_p(f) = \min_{1 \leqslant i \leqslant n}\{\nu_p(a_i)\} = 0$. thus f is primitive.

(v) For $\nu_p(f) := d$ choose $c := p^{-d} \in k^\times$. Then

$$\nu_p(c \cdot f) = \nu_p(c) + \nu_p(f) = \nu_p(p^{-d}) + d = -d + d = 0,$$

thus $c \cdot f$ is primitive. □

Proposition 8.2 (*Gauß-Lemma) For $f, g \in k[X]$ and $p \in \mathbb{P}$ we have*

$$\nu_p(f \cdot g) = \nu_p(f) + \nu_p(g).$$

proof. Write

$$f = \sum_{i=0}^{n} a_i X^i, \qquad g = \sum_{j=0}^{m} b_j X^j, \qquad f \cdot g = \sum_{k=0}^{m+n} c_k X^k, \quad c_k = \sum_{i=0}^{k} a_i b_{k-i}$$

case 1 Assume $m = 0$, i.e. $g = b_0 \in k^\times$. Then $c_k = a_k \cdot b_0$, hence

$$\nu_p(c_k) = \nu_p(a_k) + \nu_p(b_0).$$

Then we obtain

$$\nu_p(f\cdot g) \;=\; \min_{0\leqslant k\leqslant n} \nu_p(c_k) = \min_{0\leqslant k\leqslant n}\{\nu_p(a_k)+\nu_p(b_0)\} = \nu_p(b_0)+\min_{0\leqslant k\leqslant n}\{\nu_p(a_k)\} \;=\; \nu_p(g)+\nu_p(f)$$

case 2 Assume $\nu_p(f) = 0 = \nu_p(g)$, i.e. f, g are primitive. Clearly $\nu_p(fg) \geqslant 0$. We have to show: $\nu_p(fg) = 0$. Let $i_0 := \max\{i \mid \nu_p(a_i) = 0\}$ and $j_0 := \max\{j \mid \nu_p(b_j) = 0\}$. Then

$$c_{i_0+j_0} = \sum_{i=0}^{i_0+j_0} a_i b_{i_0+j_0-i} = \underbrace{\sum_{i=0}^{i_0-1} a_i b_{i_0+j_0-i}}_{(A)} + a_{i_0+j_0} + \underbrace{\sum_{i=i_0+1}^{i_0+j_0} a_i b_{i_0+j_0-i}}_{(B)}$$

We have $\nu_p(a_{i_0}b_{j_0}) = \nu_p(a_{i_0}) + \nu_p(b_{j_0}) = 0$. We have $i_0 + j_0 - i > j_0$, hence $\nu_p(b_{i_0+j_0-i}) \geqslant 1$ for $0 \leqslant i \leqslant i_0 - 1$. Then

$$\begin{aligned}
\nu_p(A) &= \nu_p\left(\sum_{i=0}^{i_0-1} a_i b_{i_0+j_0-i}\right) &\geqslant& \min_{0\leqslant i\leqslant i_0-1}\{\nu_p(a_i b_{i_0+j_0-1})\} \\
&& = & \min_{0\leqslant i\leqslant i_0-1}\{v_p(a_i) + \nu_p(b_{i_0+j_0-1})\} \\
&& \geqslant & \min_{0\leqslant i\leqslant i_0-1}\{\nu_p(b_{i_0+j_0-1})\} \\
&& \geqslant & \; 1 \\
\nu_p(B) &= \nu_p\left(\sum_{i=i_0+1}^{i_0+j_0} a_i b_{i_0+j_0-i}\right) &\geqslant& \; 1.
\end{aligned}$$

Since we have

$$0 = \nu_p(a_{i_0}b_{j_0}) \geqslant \min\{\nu_p(c_{i_0+j_0}), \nu_p(A), \nu_p(B)\} = \nu_p(c_{i_0+j_0}) = 0$$

we get $\nu_p(c_{i_0+j_0}) = 0$. Hence we obtain

$$\nu_p(fg) = \min\{\nu_p(c_i) \mid 0 \leqslant i \leqslant m+n\} = \nu_p(c_{i_0+j_0}) = 0.$$

case 3 Consider now the general case, i.e. f, g are arbitrary. Multiply f and g by suitable constants a and b, such that $\tilde{f} := af$ and $\tilde{g} := bg$ are primitive. Then by the first two cases we have

$$\begin{aligned}
\nu_p(fg) &= \nu_p\left(\frac{1}{a}\frac{1}{b}\tilde{f}\tilde{g}\right) \overset{1}{=} \nu_p\left(\frac{1}{a}\frac{1}{b}\right) + \nu_p(\tilde{f}\tilde{g}) \overset{2}{=} \nu_p\left(\frac{1}{a}\right) + \nu_p\left(\frac{1}{b}\right) + \underbrace{\nu_p(\tilde{f})}_{=0} + \underbrace{\nu_p(\tilde{g})}_{=0} \\
&= \nu_p\left(\frac{1}{a}\right) + \nu_p(\tilde{f}) + \nu_p\left(\frac{1}{b}\right) + \nu_p(\tilde{g}) = \nu_p\left(\frac{1}{a}\tilde{f}\right) + \nu_p\left(\frac{1}{b}\tilde{g}\right) \\
&= \nu_p(f) + \nu_p(g),
\end{aligned}$$

which finishes the proof. $\square$

Theorem 8.3 (*Eisenstein's criterion for irreducibility*) *Let R be a factorial domain, $p \in \mathbb{P}$ and*

$$f = \sum_{i=0}^{n} a_i X^i \quad \in R[X] \backslash \{0\}$$

Assume that f is primitive and we have

(i) $\nu_p(a_0) = 1$,

(ii) $\nu_p(a_i) \geqslant 1$ *or* $a_i = 0$ *for* $1 \leqslant i \leqslant n-1$ *and*

(iii) $\nu_p(a_n) = 0$

Then f is irreducible over $R[X]$.

proof. Assume that $f = g \cdot h$ with some $g, h \in R[X]$. Write

$$g = \sum_{i=0}^{r} b_i X^i, \qquad h = \sum_{j=0}^{s} c_i X^j, \qquad \text{with } r + s = n$$

Then we have $a_0 = b_0 c_0$. W.l.o.g. $\nu_p(b_0) = 1$ and $\nu_p(c_0) = 0$. Further $a_n = b_r c_s$, thus we must have $\nu_p(b_r) = \nu_p(c_s) = 0$ for $\nu_p(a_n) = 0$. Let now

$$d := \max\{i \mid \nu_p(b_j) \geqslant 1 \text{ for } 0 \leqslant j \leqslant i\}$$

Obviously $0 \leqslant d \leqslant r - 1$. Consider

$$a_{d+1} = \underbrace{b_{d+1} c_0}_{=:A} + \underbrace{\sum_{i=0}^{d} b_i c_{d+1-i}}_{=:B} .$$

We have

$$\nu_p(A) = \nu_p(b_{d+1}) + \nu_p(c_0) = 0 + 0 = 0,$$

$$\nu_p(B) \geqslant \min_{0 \leqslant i \leqslant d} \{\nu_p(b_i c_{d+1-1}) \geqslant 1$$

and thus $\nu_p(a_{d+1}) = 0$. But this implies $d + 1 = n \Leftrightarrow n - 1 = d \leqslant r - 1 \Rightarrow n \leqslant r \Rightarrow n = r$. Then we have $s = 0$, thus $h = c_0$ is constant. Further for $q \in \mathbb{P}$ we have

$$0 = \nu_q(f) = \nu_q(g c_o) = \underbrace{\nu_q(g)}_{\geqslant 0} + \nu_q(c_0)$$

i.e. $\nu_q(c_0) = 0$, hence $c_0 \in R^\times$ and f is irreducible. □

Theorem 8.4 (*Gauß*) *Let R be a factorial domain. Then $R[X]$ is factorial.*

proof. Let $f \in R[X] \backslash \{0\} \subseteq k[X]$ where $k = \mathrm{Quot}(R)$. Since $k[X]$ is factorial, we can write

$$f = c \cdot f_1 \cdots f_n, \quad f_i \in k[X] \text{ prime }, \ c \in k^\times$$

W.l.o.g the. f_i are primitive, otherse multiply them by suitable constants. In particular we have $f_i \in R[X]$. Note that $c \in R$: For $p \in \mathbb{P}$, we have

$$0 = \nu_p(f) = \nu_p(c) + \sum_{i=1}^{n} \nu_p(f_i) = \nu_p(c).$$

Write $c = \epsilon \cdot p_1 \cdots p_r$ with some $\epsilon \in R^\times$ and $p_i \in \mathbb{P}$.Then by

Claim (a) $f_i \in R[X]$ are prime for $1 \leqslant i \leqslant n$.

Claim (b) $p_i \in R[X]$ are prime for $1 \leqslant i \leqslant r$.

we have found a factorization of f into prime elements and hence $R[X]$ is factorial. Now prove the claims.

(a) Let $g, h \in R[X]$ such that $gh \in (f_i) = f_i R[X]$.
May assume that $g \in f_i k[X]$, i.e. $g = f_i \tilde{g}$ for some $\tilde{g} \in k[X]$. For $p \in \mathbb{P}$ we obtain

$$0 \leqslant \nu_p(g) = \underbrace{\nu_p(f_i)}_{=0} + \nu_p(\tilde{g}) = \nu_p(\tilde{g}).$$

Thus we get $\tilde{g} \in R[X]$, which implies $g = f_i \tilde{g} \in f_i R[X] = (f_i)$.

(b) Since $\pi : R \longrightarrow R/(p)$ induces a map $\psi : R[X] \longrightarrow R/(p)[X]$ with $\ker(\psi) = pR[X]$ we have

$$R[X]/pR[X] \cong R/pR[X].$$

Since R/pR is an integral domain, (p) is prime. □

Corollary 8.5 *Let k be a field. Then $k[X_1, \ldots X_n]$ is factorial for any $n \in \mathbb{N}$.*

Corollary 8.6 *Let R be a factorial domain, $k = Quot(R)$. If $f \in R[X]$ is irreducible over $R[X]$, then f is irreducible over $k[X]$.*

proof. Let $0 \neq f = c \cdot f_1 \cdots f_n$ be decomposition of f in $k[X]$, i.e. $c \in k^\times$ and $f_i \in k[X]$ irreducible for $1 \leqslant i \leqslant n$. We may assume that the f_i are primitive, hence contained in $R[X]$, since we can multiply them by suitable constants. We still have to show $c \in R$. Since $f \in k[X]$, i.e. $\nu_p(f) \geqslant 0$ we have

$$\nu_p(f) = \nu_p(c \cdot f_1 \cdots f_n) = \nu_p(c) + \sum_{i=1}^{n} \underbrace{\nu_p(f_i)}_{=0} = \nu_p(c) \overset{!}{\geqslant} 0$$

Thus $c \in R$. Then the decomposition from above is in R - but since f is irreducible in R, we have $n = 1$ and $c \in R^\times$. □

§ 9 Absolute values

Definition 9.1 Let k be a field. A map

$$|\cdot| : k \longrightarrow \mathbb{R}_{\geqslant 0}$$

is called an *absolute value*, if

(i) *positive definiteness:* $|x| = 0 \iff x = 0$

(ii) *multiplicativeness:* $|xy| = |x| \cdot |y|$ for all $x, y \in k$.

(iii) *triangle inequality:* $|x + y| \leqslant |x| + |y|$ for all $x, y \in k$.

Example 9.2 (i) The 'normal' absolute value $|\cdot|_\infty$ on $\mathbb{C}$ and on any of its subfields denotes an absolute value.

(ii) Let $\nu_k^\times \longrightarrow \mathbb{Z}$ be a discrete valuation, $\rho \in (0, 1)$. Then

$$|\cdot|_\nu : k \longrightarrow \mathbb{R},\ x \mapsto \begin{cases} \rho^{\nu(x)} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is an absolute value on k, since

(1) Trivial, since $|0| = 0$ and $\rho^x \neq 0$ for any $x \in \mathbb{Z}$.

(2) Clearly $|xy|_\nu = \rho^{\nu(xy)} = \rho^{\nu(x)+\nu(y)} = \rho^{\nu(x)}\rho^{\nu(y)} = |x|_\nu |y|_\nu$.

(3) Further

$$|x+y|_\nu = \rho^{\nu(x+y)} \leqslant \rho^{\min\{\nu(x),\nu(y)\}} = \max\{\rho^{\nu(x)}, \rho^{\nu(y)}\} = \max\{|x|_\nu, |y|_\nu\} \leqslant |x|_\nu + |y|_\nu$$

(iii) For the p-adic valuation ν_p on $\mathbb{Q}$ we choose $\rho := \frac{1}{p}$. Then $|x|_p = p^{-\nu_p(x)}$ is an absolute value.

Remark + definition 9.3 Let k be a field, $|\cdot|$ an absolute value on k.

(i) $|1| = |-1| = 1$ and $|x| = |-x|$ for all $x \in k$.

(ii) The absolute value is called trivial, if $|x| = 1$ for all $x \in k$.

proof. We have $|1| = |1 \cdot 1| = |1| \cdot |1|$, hence $|1| = 1$. Moreover $|-1| = |1 \cdot (-1)| = |1| \cdot |-1|$, hence $|-1| = 1$. For $x \in k$ we have $|-x| = |(-1) \cdot x| = |-1| \cdot |x| = |x|$. □

Proposition + definition 9.4 Let k be a field with $\mathrm{char}(k) = 0$, i.e. $k \supseteq \mathbb{Q}$ and $|\cdot|$ an absolute value on k.

(i) $|\cdot|$ is called *archimedean*, if $|n| > 1$ for all $n \in \mathbb{Z}\backslash\{-1, 0, 1\}$.

(ii) $|\cdot|$ is called *nonarchimedean*, if $|n| \leqslant 1$ for all $n \in \mathbb{Z}$.

(iii) $|\cdot|$ is either archimedean or nonarchimedean.

(iv) The p-adic absolute value on $\mathbb{Q}$ is nonarchimedean.

proof of (iii). Since $|n| = |-n|$, it suffices to check $n \in \mathbb{N}$. Let $a \in \mathbb{N} \subseteq k$ with $|a| > 1$. Assume there exists $b \in \mathbb{N}_{>1}$ with $|b| \leqslant 1$. Write

$$a = \sum_{i=0}^{N} \alpha_i b^i \qquad \alpha_i \in \{0, \ldots b-1\},\ |N| = \lfloor \log_b(a) \rfloor.$$

Then we have

$$|a| \leqslant \sum_{i=0}^{\lfloor \log_b(a) \rfloor} |\alpha_i||b|^i \leqslant \log_b(a) \cdot \max_{0 \leqslant i \leqslant \lfloor \log_b(a) \rfloor} \{|\alpha_i|\} =: \log_b(a) \cdot c,$$

$$|a^n| \leqslant \log_b(a^n) \cdot c = n \cdot \log_b(a) \cdot c$$

and $|a^n|$ grows linearly in n. Likewise we get for $n \in \mathbb{N}$

$$a^n = \sum_{i=0}^{\lfloor \log_b(a^n) \rfloor} \alpha_i^{(n)} b^i, \qquad \alpha_i^{(n)} \in \{0, \ldots b-1\},$$

$$|a^n| = |a|^n \leqslant (\log_b(a) \cdot c)^n$$

which grows exponentially in n, which is a contradiction. Hence the claim follows. □

Remark 9.5 *An absolute value $|\cdot|$ on a field k induces a metric*

$$d(x, y) := |x - y|, \qquad x, y \in k$$

Therefore, k as a topology and aspects as 'convergence' and 'cauchy sequences' are meaningful.

Definition + remark 9.6 (i) Two absolute values $|\cdot|_1, |\cdot|_2$ on k are called *equivalent*, if there exists $s \in \mathbb{R}$, such that $|x|_1 = |x|_2^s$ for all $x \in k$. In this case, we write $|\cdot|_1 \sim |\cdot|_2$.

(ii) Two absolutes values $|\cdot|_1, |\cdot|_2$ are equivalent if and only if the induce the same topology on k.

proof. Is left for the reader as an exercise.

Example 9.7 The p-adic absolute values on $\mathbb{Q}$ are not equivalent for $p \neq q \in \mathbb{P}$. Consider

$$|p^n|_p = p^{-n} \xrightarrow{n \to \infty} 0, \qquad |p^n|_q = 1 \text{ for all } n \in \mathbb{N}$$

Moreover we have $|\cdot|p \not\sim |\cdot|_\infty$, since by the transittivity of equivalence of absolute values, we have

$$|\cdot|_p \sim |\cdot|_\infty \sim |\cdot|_q$$

which is not true.

Theorem 9.8 (*Ostrowski*) *Any nontrivial absolute value $|\cdot|$ on $\mathbb{Q}$ is equivalent either to the standard absolute value $|\cdot|_\infty$ on $\mathbb{Q}$ or to a p-adic absolute value $|\cdot|_p$ for some $p \in \mathbb{P}$.*

proof. **case 1** Assume $|\cdot|$ is nonarchimedean. We want to show, that in this case $|\cdot| \sim |\cdot|_p$ for some $p \in \mathbb{P}$. Since $|\cdot|$ is non-trivial, there exists $x \in \mathbb{N}$ such that

$$|x| = \left|\prod_{p\in\mathbb{P}} p^{\nu_p(x)}\right| = \prod_{p\in\mathbb{P}} |p|^{\nu_p(x)} \neq 1$$

for at least one $x \in \mathbb{Q}$, hence, we have $|p| \neq 1$ for at least one $p \in \mathbb{P}$, i.e. $|p| < 1$. Assume there is another prime $q \neq p$ with $|q| < 1$. Then we find $N \in \mathbb{N}$, such that

$$|p|^N \lneq \frac{1}{2}, \qquad |q|^N \lneq \frac{1}{2}.$$

Moreover, since p^N, q^N are coprime, we can write

$$1 = a \cdot p^N + b \cdot q^N \qquad \text{for suitable } a, b \in \mathbb{Z}.$$

So the contradiction follows by

$$1 = |1| = \left|ap^N + bq^N\right| \leqslant \underbrace{|a|}_{\leqslant 1} \underbrace{\left|p^N\right|}_{<\frac{1}{2}} + \underbrace{|b|}_{\leqslant 1} \underbrace{\left|q^N\right|}_{<\frac{1}{2}} < 1,$$

hence we have $|q| = 1$ for any $q \neq p \in \mathbb{P}$. Let now $s := -\log_p |p|$. For $x \in \mathbb{Q}^\times$ we obtain

$$|x| = \left|\prod_{\tilde{p}\in\mathbb{P}} \tilde{p}^{\nu_{\tilde{p}}(x)}\right| = \prod_{\tilde{p}\in\mathbb{P}} |\tilde{p}|^{\nu_{\tilde{p}}(x)} = |p|^{\nu_p(x)} = p^{-s\cdot\nu_p(x)} = \left(p^{-\nu_p(x)}\right)^s = |x|_p^s$$

thus we have $|\cdot| \sim |\cdot|_p$.

case 2 Let now $|\cdot|$ be archimedean. We now have to show $|\cdot| \sim |\cdot|_\infty$. For $n \in \mathbb{N}_{\geqslant 2}$ we have

$$1 < |n| = \left|\sum_{i=1}^{n} 1\right| \leqslant \sum_{i=1}^{n} |1| = n.$$

For any $a \in \mathbb{N}_{\geqslant 2}$ we find $s := s(a) \in \mathbb{R}_{<0}$ such that

$$|a| = |a|_\infty^s = a^s$$

namely

$$s = \log_a(|a|) = \frac{\log(|a|)}{\log(a)}.$$

Claim (a) We have

$$\frac{\log(|a|)}{\log(a)} = \frac{\log(|2|)}{\log(2)}.$$

Since now s is independent of a, we have $|\cdot| \sim |\cdot|_\infty$. Prove now the claim:

(a) For $n \in \mathbb{N}$ write

$$2^n = \sum_{i=0}^{N} \alpha_i a^i \quad \text{with } \alpha_i \in \{0, \dots a-1\} \text{ and } N \leqslant \log_a 2^n = n \cdot \frac{\log(2)}{\log(a)}.$$

Then we have

$$|2|^n = |2^n| \leqslant \sum_{i=0}^{N} \underbrace{|\alpha_i|}_{\leqslant \alpha_i < a} \overbrace{|a|^i} \leqslant |a|^N \leqslant (N+1)^{\cdot} a \cdot |a|^N,$$

hence we get

$$\begin{aligned} n \cdot \log(|2|) &\leqslant \log(N+1) + \log(a) + N \log(|a|) \\ &\leqslant \log\left(n \cdot \frac{\log(2)}{\log(a)} + 1\right) + \log(a) + n \cdot \frac{\log(2)}{\log(a)} \cdot \log(|a|). \end{aligned}$$

Multiplying the equation by $\frac{1}{n} \cdot \frac{1}{\log(2)}$ gives us

$$\frac{\log(|2|)}{\log(2)} \leqslant \frac{1}{n} \cdot \log\left(n \cdot \frac{\log(2)}{\log(a)} + 1\right) + \frac{\log(|a|)}{\log(a)}$$

and thus

$$\frac{\log(|2|)}{\log(2)} \leqslant \frac{\log(|a|)}{\log(a)}.$$

Swapping the roles of a and 2 in the equation above gives us the other inequality. Hence we have equality, which proves the claim. $\square$

Proposition 9.9 *Let $|\cdot|$ be a nonarchimedean absolute value on a field k.*

(i) $|x+y| \leqslant \max\{|x|, |y|\}$ *for all* $x, y \in k$.

(ii) If $|x| \neq |y|$, *then equality holds in (i).*

proof. (i) If $x = 0$, we have $|y + x| = |y| \leqslant \max\{0, |y|\} = \max\{|x|, |y|\}$. Thus assume $x \neq 0$. We have $|x + y| = |x|\left|1 + \frac{y}{x}\right|$. It suffices to show $|x+1| \leqslant \max\{1, |x|\}$. Then we get

$$|x+y| = |y| \cdot \left|1 + \frac{x}{y}\right| \leqslant |y| \cdot \max\left\{\left|\frac{x}{y}\right|, |1|\right\} \leqslant \max\{|x|, |y|\}$$

For $n \in \mathbb{N}$ we have

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Then we have

$$|x+1|^n = |(x+1)^n| = \left|\sum_{k=0}^{n}\binom{n}{k}x^k\right| \leqslant \sum_{k=0}^{n}\underbrace{\left|\binom{n}{k}\right|}_{\leqslant 1}\underbrace{|x|}_{\leqslant 1}{}^{k} \leqslant n+1,$$

hence

$$|x+1| \leqslant \sqrt[n]{n+1} \qquad \text{for all } n \in \mathbb{N}.$$

Thus $|1+x| \leqslant 1$. Since we clearly have $|x+1| \leqslant |x|$, we all in all have

$$|x+1| \leqslant \max|\{|x|, 1\}.$$

(ii) Let $z = x+y$ and assume $|x| < |y|$. We have to show $|z| = |y|$. Assume $|z| < |y|$. Then

$$|y| = |z-x| \overset{(i)}{\leqslant} \max\{|z|, |-x|\} < |y| \quad ↯$$

and the proof is done. □

Proposition 9.10 *Let $|\cdot|$ be an a nonarchimedean absolute value on a field k. Then*

(i) We have a local ring

$$\overline{\mathcal{B}}_1(0) := \{x \in k \big| |x| \leqslant 1\} =: \mathcal{O}_k$$

with maximal ideal

$$\mathcal{B}_1(0) := \{x \in k \big| |x| < 1\} =: \mathfrak{m}_k$$

(ii) Every point in ball is its center.

(iii) Balls are either disjoint or one of them is contained in the other one.

(iv) All triangles are isosceles.

proof. (i) By 9.8(i), $\mathcal{B}_1(0)$ is closed under Addition. The remaining is calculating.

(ii) Let $z \in \overline{\mathcal{B}}_r(x)$. To show: $\overline{\mathcal{B}}_r(z) = \overline{\mathcal{B}}_r(x)$.

'$\subseteq$' Let $y \in \overline{\mathcal{B}}_r(z)$, i.e. we have $|y-z| \leqslant r$. Then

$$|y-x| = |y-z+z-x| \leqslant \max\{|y-z|, |z-x|\} \leqslant r \quad \Rightarrow \quad y \in \overline{\mathcal{B}}_r(x).$$

Thus we have $\overline{\mathcal{B}}_r(z) \subseteq \overline{\mathcal{B}}_r(x)$.

'$\supseteq$' Follows by symmetry.

(iii) Let $\mathcal{B} := \overline{\mathcal{B}}_r(x)$, $\mathcal{B}' := \overline{\mathcal{B}}_{r'}(x')$ and $y \in \mathcal{B} \cap \mathcal{B}'$. W.l.o.g. $r \leqslant r'$.

Then for $z \in \mathcal{B}$ we have

$$|z-x'| = |z-x+x-y+y-x'| \leqslant \max\{|z-x|, |x-y|, |y-x'|\} = \max\{r, r, r'\} = r'$$

which implies $z \in \mathbb{B}'$. Hence we have $\mathcal{B} \subseteq \mathcal{B}'$.

(iv) Follows from 9.8(ii). $\square$

Corollary 9.11 *Let k be a field, $|\cdot|$ a nonarchimedean absolute value on k.*

(i) All balls are closed and open, considering the topology on k induced by the metric $d(x,y) = |x-y|$.

(ii) k is totally disconnected, i.e. no subset of k containing more than on element is connected.

proof. (i) Let $\mathcal{B} := \overline{\mathcal{B}}_r(x)$ be a closed ball for some $x \in k$, $r \in \mathbb{R}_{\geqslant 0}$. Then $\mathcal{B}$ topologically clearly is closed . Let now $y \in \mathcal{B}$. Then $\mathcal{B}_r(y) \subseteq \mathcal{B}$ by 9.9(ii), i.e. $\mathcal{B}$ is open.

Let now $\mathcal{B} := \mathcal{B}_r(x)$ be an open ball and $y \in k$ a boundary point. Thus for all $s > 0$ we find $z \in \mathcal{B}_s(x) \cap \mathcal{B}_r(x)$. Choose $s \leqslant r$. Then

$$d(x,y) \leqslant \max\{d(y,z), d(x,z)\} < \max\{s,r\} = r.$$

Thus $y \in \mathcal{B}_r(x)$, hence $\mathcal{B}_r(x)$ is contains its boundary and is closed.

(ii) Let $X \subseteq k$ be a subset with $x \neq y \in X$. Then for $r := |x-y| > 0$ we get

$$X = \left(\overline{\mathcal{B}}_{\frac{r}{2}}(x) \cap X\right) \cup \left(X \backslash \overline{\mathcal{B}}_{\frac{r}{2}}(x)\right)$$

which is a decomposition of X into two nonempty, disjoint open subset, i.e. the claim follows.

Example 9.12 (*Geometry on* $(\mathbb{Q}, |\cdot|_p)$*)* The unit disc in $(\mathbb{Q}, |\cdot|_p)$ is

$$\left\{\frac{a}{b} \in \mathbb{Q} \mid p \nmid b\right\} =: \mathbb{Z}_{(p)}$$

The maximal ideal is

$$\left\{\frac{a}{b} \in \mathbb{Q} \mid p \nmid b, p \mid a\right\} = p \cdot \mathbb{Z}_{(p)} = \overline{\mathcal{B}}_{\frac{1}{p}}(0)$$

We have

$$\left\{x \in \mathbb{Q} \mid |x|_p < 1\right\} = \left\{x \in \mathbb{Q} \mid |x|_\infty < \frac{1}{p}\right\}$$

Moreover

$$\mathbb{Z}_{(p)} \Big/ p\mathbb{Z}_{(p)} \cong \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p = \{\overline{0}, \overline{1}, \ldots, \overline{p-1}\}$$

$\overline{\mathcal{B}}_1(0)$ is the disjoint union of the $\overline{\mathcal{B}}_{\frac{1}{p}}(i)$ for $0 \leqslant i \leqslant p-1$, where $\overline{\mathcal{B}}_{\frac{1}{p}}(i) = i + p\mathbb{Z}_{(p)}$.

§ 10 Completions, p-adic numbers and Hensel's Lemma

Remark 10.1 *Let $|\cdot|$ be an absolute value on a field k. Let*

$$\mathcal{C} := \{(a_n)_{n\in\mathbb{N}} \mid (a_n) \text{ is Cauchy sequence in } (k, |\cdot|)\}$$

be th ring (!) of Cauchy sequences in k and

$$\mathcal{N} := \left\{(a_n)_{n\in\mathbb{N}} \mid \lim_{n\to\infty} a_n = 0\right\} \trianglelefteq \mathcal{C}$$

the ideal (!) of Cauchy sequences converging to 0. Then

(i) $\mathcal{N}$ *is a maximal ideal.*

(ii) $k' := \mathcal{C}/\mathcal{N}$ *is a field extension of k.*

(iii) $|\overline{(a_n)_{n\in\mathbb{N}}}| := \lim_{n\to\infty}(a_n) \in \mathbb{R}_{\geqslant 0}$ *is an absolute value on k' extending $|\cdot|$.*

(iv) k' *is complete with respect to $|\cdot|$.*

Remark 10.2 *If $|\cdot|$ is nonarchimedean, for every Cauchy sequence $(a_n)_{n\in\mathbb{N}} \notin \mathcal{N}$ we have $|a_m| = |a_n|$ for all $m, n \gg 0$.*

proof. Since $(a_n) \notin \mathcal{N}$, 0 is not an accumulation point of (a_n). $\Longrightarrow |a_n| \geqslant \epsilon$ for some $\epsilon > 0$ and all $n \geqslant n_0(\epsilon) =: n_0$. Thus for $n, m \geqslant n_0$ we have $|a_n - a_m| < \epsilon$. This implies by 9.8 (ii)

$$|a_n - a_m| \lneqq \max\{|a_n|, |a_m|\} \implies |a_n| = |a_m|,$$

which was the claim. □

Definition 10.3 Let $k = \mathbb{Q}$, $|\cdot| = |\cdot|_p$ for some $p \in \mathbb{P}$. Then the field k' on 10.1 is called the field of *p-adic numbers* and denoted by $\mathbb{Q}_p$. The valuation ring is called the ring of *p-adic integers* and is denoted by $\mathbb{Z}_p$.

Remark 10.4 *(i)* $\mathbb{Z} \subset \mathbb{Z}_{(p)} \subset \mathbb{Z}_p$.

(ii) The maximal ideal in $\mathbb{Z}_p$ is $p\mathbb{Z}_p$.

(iii) $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$.

(iv) $\mathbb{Z}_p$ *is a discrete valuation ring.*

proof. (i) The first inclusion is clear. For the second one consider $x = \frac{r}{s} \in \mathbb{Z}_{(p)}$. Then by definition of localization we have $p \nmid s$ and hence

$$|x| = \left|\frac{r}{s}\right| = \frac{|r|}{|s|} = |r| \leqslant 1$$

and thus $x \in \mathbb{Z}_p$. Now prove that $\mathbb{Z}$ is dence in $\mathbb{Z}_p$: Let $x \in \mathbb{Z}_p$ with p-adic expansion

$$x = \sum_{i=0}^{\infty} a_i p^i, \qquad a_i \in \{0, 1, \dots, p-1\}.$$

Define a sequence $(x_n)_{n \in \mathbb{N}}$ by

$$x_n := \sum_{i=0}^{n} a_i p^i \in \mathbb{Z}.$$

Then we have

$$|x - x_n| = \Big| \sum_{i=n+1}^{\infty} \Big| = \max_{i \geqslant n+1} \{|p^i|\} = \left|p^{n+1}\right| = p^{-(n+1)} \xrightarrow{n \to \infty} 0$$

and hence $\mathbb{Z}$ is dence in $\mathbb{Z}_p$.

(ii) Recall that the maximal ideal is given by

$$\mathfrak{m} = \{x \in \mathbb{Z}_p \mid |x| < 1\} \overset{!}{=} p\mathbb{Z}_p$$

'$\subseteq$' Let $x \in \mathfrak{m}$, i.e. $|x| < 1$. Thus we have $|x| < \left|\frac{1}{p}\right|$. This implies

$$|p^{-1}x| \leqslant 1 \iff p^{-1}x \in \mathbb{Z}_p.$$

and thus $p^{-1}x = y$ for some $y \in \mathbb{Z}_p$. Then we have $x = py \in p\mathbb{Z}_p$.

'$\supseteq$' Let $x \in p\mathbb{Z}_p$, i.e. we can write $x = py$ for some $y \in \mathbb{Z}_p$. Then $|x| = |py| = |p||y| < 1$ and hence $x \in \mathfrak{m}$.

(iii) Consider the surjective homomorphism

$$\psi_p : \mathbb{Z}_p \longrightarrow \mathbb{Z}/p\mathbb{Z}, \quad x = \sum_{i=0}^{n} a_i p^i \mapsto a_0.$$

We have

$$\ker(\psi_p) = \{x \in \mathbb{Z}_p \mid a_0 \equiv 0 \mod p\} = p\mathbb{Z}_p,$$

thus we get $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}$ by homomorphism theorem.

(iv) The absolute value $|\cdot| = |\cdot|_p$ on $\mathbb{Q}_p$ induces a discrete valuation ν on $\mathbb{Q}_p^\times$. With respect to this valuation we have

$$\mathcal{O}_\nu = \{x \in \mathbb{Q}_p \mid \nu(x) \geqslant 0\} \cup \{0\} = \{x \in \mathbb{Q}_p \mid |x| \leqslant 1\} = \mathbb{Z}_p,$$

which finishes the proof. $\square$

Proposition 10.5 *(i) Any $x \in \mathbb{Z}_p$ can uniquely be written in the form*

$$x = \sum_{i=0}^{\infty} a_i p^i, \qquad a_i \in \{0, 1, \dots, p-1\}.$$

(ii) Any $x \in \mathbb{Q}_p$ can uniquely be written in the form

$$x = \sum_{i=-m}^{\infty} a_i p^i, \qquad m \in \mathbb{Z},\ a_i \in \{0, 1, \dots, p-1\},\ a_m \neq 0.$$

proof. (i) We first obtain, that any series

$$\sum_{i=0}^{\infty} a_i p^i, \qquad a_i \in \{0, \dots, p-1\}$$

converges, since for $n > m$ we have

$$\left|\sum_{i=0}^{n} a_i p^i - \sum_{i=0}^{m} a_i p^i\right| = \left|\sum_{i=n+1}^{m} a_i p^i\right| = \left|p^{m+1}\right| \underbrace{\left|\sum_{i=n+1}^{m} a_i p^{i-(m+1)}\right|}_{\leqslant 1} \leqslant \left|p^{m+1}\right|.$$

uniqueness Let

$$x = \sum_{i=0}^{\infty} a_i p^i = \sum_{i=0}^{\infty} b_i p^i, \qquad a_i, b_i \in \{0, 1, \dots, p-1\}$$

representations of $x \in \mathbb{Q}_p$. Assume them to be different and define $i_o := \min\{i \in \mathbb{N}_0 \mid a_i \neq b_i\}$. Then

$$0 = \left|\sum_{i=0}^{\infty} a_i p^i - \sum_{i=0}^{\infty} b_i p^i\right| = \left|\underbrace{p^{i_0}(a_{i_0} - b_{i_0})}_{=:A} + p^{i_0+1} \cdot \underbrace{\left(\sum_{i=i_0+1}^{\infty} a_i p^{i-(i_0+1)} - \sum_{i=i_0+1}^{\infty} b_i p^{i-(i_0+1)}\right)}_{=:B}\right|.$$

We obtain $\nu_p(A) = p^{-i_0}$ and

$$B \in \mathbb{Z}_p, \quad \nu_p\left(p^{i_0+1} \cdot B\right) = \nu_p\left(p^{i_0+1}\right) \underbrace{\nu_p(B)}_{\leqslant 1} \leqslant \nu_p\left(p^{i_0+1}\right) = p^{-(i_0+1)},$$

so all in all

$$0 = \left|A + p^{i_0+1} \cdot B\right| \overset{9.8(ii)}{=} \max\{p^{-i_0}, p^{-(i_0+1)}\} = p^{-i_0} \ \lightning.$$

existence Look at $\overline{x} \in \mathbb{Z}_p / p\mathbb{Z}_p = \mathbb{F}_p$.

Let a_0 be the representative of x in $\{0, 1, \dots, p-1\}$. Then we have

$$|x - a_0| < 1 \ \Leftrightarrow\ |x - a_0| \leqslant \frac{1}{p}.$$

In the next step, let a_1 be the representative of $\frac{1}{p}(x-a_0)$ in $\{0,1,\ldots,p-1\}$. Then

$$\left|\frac{1}{p}(x-a_0)-a_1\right| = \left|\frac{1}{p}\right||x-a_0-a_1p| \leqslant \frac{1}{p}$$

and thus $|x-a_0-a_1p| \leqslant \frac{1}{p^2}$. Inductively we let a_n be the representative of

$$\frac{1}{p^n}(x-a_0-a_1p-\ldots-a_{n-1}p^{n-1}) = \frac{1}{p^n}\left(x-\sum_{i=0}^{n-1}a_ip^i\right)$$

in $\{0,1,\ldots,p-1\}$. Then we have

$$\left|x-\sum_{i=0}^{n-1}a_ip^i\right| \leqslant \frac{1}{p^{n+1}}.$$

and finally

$$\lim_{n\to\infty}\left|x-\sum_{i=0}^{n-1}a_ip^i\right| \leqslant \lim_{n\to\infty}\frac{1}{p^{n+1}} = 0 \implies x = \sum_{i=0}^{\infty}a_ip^i.$$

(ii) If $|x| = p^m$ for some $m \in \mathbb{Z}$, we have

$$|x\cdot p^m| = |d|\cdot|p^m| = p^m\cdot p^{-m} = 1, \qquad \text{i.e. } x\cdot p^m \in \mathbb{Z}_p^\times$$

By part (i) we conclude

$$x\cdot p^m = \sum_{i=0}^{\infty}a_ip^i, \quad a_0 \neq 0.$$

Thus we have

$$x = \frac{1}{p^m}\cdot x\cdot p^m = \frac{1}{p^m}\cdot\sum_{i=0}^{\infty}a_ip^i = \sum_{i=-m}^{\infty}a_{i+m}p^i,$$

which was the assertion. □

Remark 10.6 *What is -1 in $\mathbb{Q}_p$? We have $a_0 = p-1$, since $\overline{p-1}-\overline{(-a)} = \overline{p} = 0$. a_1 is the representative of $\frac{1}{p}(-1-(p-1)) = -1$, i.e. $a_1 = p-1$. a_2 is the representative of $\frac{1}{p^2}(-1-(p-1)-(p-1)p) = -1$, i.e. $a_2 = p-1$. Inductively we have $a_n = p-1$ for all $n \in \mathbb{N}_0$, so we get*

$$-1 = \sum_{i=0}^{\infty}a_ip^i = \sum_{i=0}^{\infty}(p-1)p^i.$$

Moreover we obtain

$$\sum_{i=0}^{\infty}(p-1)p^i = (p-1)\sum_{i=0}^{\infty}p^i = (p-1)\cdot\frac{1}{1-p} = \frac{p-1}{1-p} = -1.$$

Remark 10.7 *Let*

$$x = \sum_{i=0}^{\infty} a_i p^i, \qquad y = \sum_{i=0}^{\infty} b_i p^i$$

p-adic integers. Then

$$x + y = \sum_{i=0}^{\infty} c_i p^i$$

with coefficients

$$c_0 = \begin{cases} a_0 + b_0 & \text{if } a_0 + b_0 < p \\ a_0 + b_0 - p & \text{if } a_0 + b_0 \geqslant p \end{cases}$$

$$c_1 = \begin{cases} a_1 + b_1 & \text{if } a_0 + b_0 < p \text{ and } a_1 + b_1 < p \\ a_1 + b_1 - p & \text{if } a_0 + b_0 < p \text{ and } a_1 + b_1 \geqslant p \\ a_1 + b_1 + 1 & \text{if } a_0 + b_0 \geqslant p \text{ and } a_1 + b_1 + 1 < p \\ a_1 + b_1 + 1 - p & \text{if } a_0 + b_0 \geqslant p \text{ and } a_1 + b_1 + 1 \geqslant p \end{cases}$$

Inductively let

$$\epsilon_0 := 0, \qquad \epsilon_i := \begin{cases} 0 & \text{if } a_i + b_i + \epsilon_{i-1} < p \\ 1 & \text{if } a_i + b_i + \epsilon_{i-1} \geqslant p \end{cases} \qquad \text{for } i \geqslant 1$$

Then we have

$$c_i = \begin{cases} a_i + b_i + \epsilon_i & \text{if } a_i + b_i + \epsilon_i < p \\ a_i + b_i + \epsilon_i - p & \text{if } a_i + b_i + \epsilon_i \geqslant p \end{cases}$$

Remark 10.8 *(i)* $\sqrt{p} \notin \mathbb{Q}_p$*, since* $|\sqrt{p}| = \sqrt{|p|} = \sqrt{\frac{1}{p}} \in \left(\frac{1}{p}, 1\right)$*, which is not possible.*

(ii) Let $a \in \mathbb{Z}_p^\times$ *with image* $\overline{a} \in \mathbb{F}_p^\times \backslash \mathbb{F}_p^{\times^2}$*, where*

$$\mathbb{F}_p^{\times^2} = \{x \in \mathbb{F}_p \mid \text{ there exists } y \in \mathbb{F}_p : y^2 = x\}$$

denotes the set of squares. Then $\sqrt{a} \notin \mathbb{Q}_p$*. Assume* a *is a aquare, i.e.* $b^2 = a$*. Then*

$$|b| = \sqrt{|a|} = 1 \quad \Rightarrow \quad b \in \mathbb{Z}_p^\times$$

But then $\overline{b} \in \mathbb{F}_p$ *satisfies* $\overline{b}^2 \equiv a$*, which is a contradiction, since* $a \notin \mathbb{F}_p^{\times^2}$*.*

(iii) Let now $\overline{\mathbb{Q}}_p$ *be the algebraic closure of* $\mathbb{Q}_p$ *with valuation ring* $\overline{\mathbb{Z}}_p$ *and maximal ideal* $\overline{\mathfrak{m}}_p$*. Then* $\overline{\mathbb{Z}}_p / \overline{\mathfrak{m}}$ *is algebraically closed. Moreover* $\mathbb{Q}_p$ *is complete with respect to* $|\cdot|_p$*. The completion* $\mathbb{C}_p$ *of* $\overline{\mathbb{Q}}_p$ *is complete and algebraically closed, but:*

(1) $|\cdot|_p$ *is not a discrete valuation.*

(2) $\overline{\mathbb{Z}}_p$ *is not a discrete valuation ring.*

(3) $\overline{\mathfrak{m}}_p$ *is not a principal ideal.*

Theorem 10.9 (*Hensel's Lemma*) *Let*

$$f = \sum_{i=0}^{n} a_i X^i \in \mathbb{Z}_p[X], \qquad \overline{f} = \sum_{i=0}^{n} \overline{a_i} X^i \in \mathbb{F}[X]$$

where $\overline{f}$ is the reduction of f in $\mathbb{F}[X]$. Suppose that $\overline{f} = f_1 \cdot f_2$ with $f_1, f_2 \in \mathbb{F}_p[X]$ relatively prime. Then there exist $g, h \in \mathbb{Z}_p[X]$, such that

$$f = g \cdot h, \quad \overline{g} = f_1, \overline{h} = f_2, \quad \deg(f_1) = \deg(g)$$

proof. Let $d := \deg(f), m := \deg(f_1)$. Then $\deg(f_2) \leqslant d - m$. Choose $g_0, h_0 \in \mathbb{Z}_p[X]$ such that $\overline{g_0} = f_1, \overline{h_0} = f_2, \deg(g_0) = m, \deg(h_0) = d - m$. *Strategy:* Find $g_1 = g_0 + pc_1$, $h_1 = h_0 + pd_1$ with some $c_1, d_1 \in \mathbb{Z}_p[X]$, such that

$$f - g_1 h_1 \quad \in p^2 \mathbb{Z}_p[X].$$

Therefore we have a

Claim (a) For $n \geqslant 1$ there exists $c_n, d_n \in \mathbb{Z}_p[X]$ with $\deg(c_n) \leqslant m, \deg(d_n) \leqslant d - m$ and

$$f - g_n h_n \in p^{n+1} \mathbb{Z}_p[X], \qquad \text{where } g_n = g_{n-1} + p^n c_n, \quad h_n = h_{n-1} + p^n d_n.$$

Assuming (a), write

$$g_n = \sum_{i=0}^{m} g_{n,i} X^i, \qquad h_n = \sum_{i=0}^{d-m} h_{n,i} X^i.$$

By construction, the $(g_{n,i})$ converge to some $\alpha_i \in \mathbb{Z}_p$ and the $(h_{n,i})$ converge to some $\beta_i \in \mathbb{Z}_p$. Let

$$g := \sum_{i=0}^{m} \alpha_i X^i, \qquad h := \sum_{i=0}^{d-m} \beta_i X^i.$$

Observe, that $\deg(g) = m, \deg(h) = d - m$. Obviously we have

$$f = g \cdot h.$$

It remains to show the claim.

(a) c_n, d_n have to satisfy

$$\begin{aligned} f - g_n h_n &= f - (g_{n-1} + p^n c_n) \cdot (h_{n-1} + p^n d_n) \\ &= f - g_{n-1} h_{n-1} - p^n \cdot (g_{n-1} d_n + h_{n-1} c_n + p^n c_n d_n) \\ &\overset{!}{\in} p^{n+1} \mathbb{Z}_p[X] \end{aligned}$$

where $f - g_{n-1}h_{n-1} \in p^n\mathbb{Z}_p[X]$ by hypothesis. We get

$$\tilde{f}_n := \frac{1}{p^n}(f - g_{n-1}h_{n-1}) \equiv c_n h_{n-1} + d_n g_{n-1} \mod p \ (*)$$

Since f_1, f_2 are relatively prime and $g_j \equiv g_k \mod p$ for any j, k, we find integers $a, b \in \mathbb{Z}$, such that

$$af_1, bf_2 = 1 \quad \Longrightarrow \quad ag_{n-1} + bh_{n-1} \equiv 1 \mod p.$$

Multiplying the equation by $\tilde{f}_n$ gives us

$$\tilde{f}_n \equiv \underbrace{a\tilde{f}_n}_{=:\tilde{d}_n} g_{n-1} + \underbrace{b\tilde{f}_n}_{=:\tilde{c}_n} h_{n-1} \mod p \ (**).$$

Further $\mathbb{Z}_p[X]$ is euclidean, thus we can choose $q_n, r_n \in \mathbb{Z}_p[X]$, $\deg(r_n) < m$ such that

$$b\tilde{f}_n = q_n g_{n-1} + r_n.$$

By $(**)$ we have

$$g_{n-1}\left(a\tilde{f}_n + q_n h_{n-1}\right) + r_n \equiv \tilde{f}_n \mod p.$$

Let now $c_n = r_n, d_n = a\tilde{f}_n + q_n h_{n-1}$. All the terms are divisible by p. Then

$$d_n \equiv a\tilde{f}_n + q_n h_{n-1} \mod p.$$

Thus $(*)$ holds and we have

$$\deg(d_n) = \deg(\overline{d_n}) \leqslant \deg\left(\underbrace{\overbrace{\overline{\tilde{f}}_n}^{\leqslant d} - \overbrace{\overline{c}_n}^{<m}\ \overbrace{\overline{h}_{n-1}}^{<d-m}}_{\leqslant d}\right) - \underbrace{\deg(\overline{g}_{n-1})}_{=m} \leqslant d - m$$

since $\overline{d}_n\overline{g}_{n-1} = \overline{\tilde{f}}_n - \overline{c}_n\overline{h}_{n-1}$. Thus, the claim is proved. □

Corollary 10.10 *Let $p \in \mathbb{P}$ odd. Then $a \in \mathbb{Z}_p^\times$ is a square if and only if $\overline{a} \in \mathbb{F}_p^\times$ is a square.*

Proposition 10.11 *$a \in \mathbb{Q}$ is a square if and only if $a > 0$ and a is a square in $\mathbb{Q}_p$ for all $p \in \mathbb{P}$.*
Remark: This is a special case of the 'Hasse-Minkowski-Theorem'.

Kapitel III

Rings and modules

§ 11 Multilinear Algebra

In this section, R will always be a commutative, unitary ring.

Reminder 11.1 (i) An *R-module* is an abelian group $(M, +)$ together with a scalar multiplication

$$\cdot : R \times M \longrightarrow M$$

with the usual properties of a vector space, i.e. for any $m, n \in M, r, s \in R$ we have

(1) $r \cdot (s \cdot m) = (rs) \cdot m$

(2) $(r + s) \cdot m = r \cdot m + s \cdot m$

(3) $r \cdot (m + n) = r \cdot m + r \cdot n$

(4) $1_R \cdot m = m$

(ii) A map $\phi : M \longrightarrow M'$ of R-modules M, M' is called *R-linear* or *R-module homomorphism*, if

$$\phi(r \cdot m + s \cdot n) = r \cdot \phi(m) + s \cdot \phi(n) \qquad \text{for all } r, s \in R, m, n \in M.$$

(iii) A subset $S \subseteq M$ of an R-module is called an *R-submodule of M*, if S is an R-module.

(iv) R itself is an R-module, the submodules are the ideals of R.

(v) If $\phi : M \longrightarrow M'$ is R-linear, then

$$\ker(\phi) = \{m \in M \mid \phi(m) = 0\},$$

$$\operatorname{im}(\phi) = \{m' \in M' \mid \phi(m) = m' \text{ for some } m \in M\}$$

are R-submodules.

(vi) If $M \subseteq M'$ is a submodule, then the factor group M/M' is an R-module via

$$a \cdot \overline{m} = \overline{a \cdot m}.$$

(vii) For an R-linear map $\phi : M \longrightarrow M''$, we have

$$\operatorname{im}(\phi) \cong M / \ker(\phi).$$

(viii) An R-module M is called *free*, if there exists a subset $X \subseteq M$, such that every $m \in M$ has a unique representation

$$m = \sum_{x \in X} a_x \cdot x, \qquad a_x \in R,\ a_x \neq 0 \text{ only for finitely many } x \in X.$$

In this case, X is called the rank of M.

(ix) Not every R-module is free: Indeed let $0 \lneq I \lneq R$ be a proper ideal. Then R/I is not free: Let $X \subseteq R$, such that $\overline{X} \subseteq R/I$ generates the R-module R/I. Let $x \in X$ and $a \in I\backslash\{0\}$. Then we have

$$x \cdot \overline{x} = \overline{a \cdot x} = \overline{0} = \overline{0 \cdot x} = 0 \cdot \overline{x},$$

hence we have found two different reapersentations of 0. Thus R/I is not free.

(x) For any $n \in \mathbb{N}$, $n\mathbb{Z}$ is a free module

(xi) If $I \leqslant R$ is not a principle ideal, then I is not a free R-module., since for $x, y \in I$ with $y \notin (x)$ we have $xy - yx = 0$. Again we have a nontrivial representation of 0 and I is not free.

Definition + proposition 11.2 Let R be a ring, M, M' R-modules.

(i) The set of R-module homomorphisms

$$\operatorname{Hom}_R(M, M') = \{\phi : M \longrightarrow M' \mid \phi \text{ is } R\text{-linear }\}$$

is again an R-module.

(ii) $M^* = \operatorname{Hom}_R(M, R)$ is called the *dual module* of M.

Let now

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

be a short exact sequence of R-modules M, M', M'', i.e. α is injective and β is surjective.

(iii) Then we have a short exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Hom}_R(N, M') & \xrightarrow{\alpha_*} & \operatorname{Hom}_R(N, M) & \xrightarrow{\beta_*} & \operatorname{Hom}_R(N, M'') \\ & & \phi & \mapsto & \alpha \circ \phi, \quad \psi & \mapsto & \beta \circ \psi \end{array}$$

(iv) We have s short exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Hom}_R(M'', N) & \xrightarrow{\beta^*} & \operatorname{Hom}_R(M, N) & \xrightarrow{\alpha^*} & \operatorname{Hom}_R(M', N) \\ & & \phi & \mapsto & \phi \circ \beta, \quad \psi & \mapsto & \psi \circ \alpha \end{array}$$

(v) N is called a *projective* module, if β_* is surjective for all short exact sequences as in (iii).

(vi) N is called an *injective* module, if α^* is surjective for all short exact sequences an in (iv).

proof. (i) This is clear: For all $\phi, \phi_1, \phi_2 \in \mathrm{Hom}_R(M, M')$ and $a \in R$ we have

$$(\phi_1 + \phi_2)(x) = \phi_1(x) + \phi_2(x), \qquad (a \cdot \phi)(x) = a \cdot \phi(x)$$

(iii) α_* is R-linear: For any $\phi_1, \phi_2 \in \mathrm{Hom}_R(N, M')$ and $x \in N$ we have

$$\alpha_*(\phi_1 + \phi_2)(x) = (\alpha \circ (\phi_1 + \phi_2))(x) = \alpha(\phi_1(x) + \phi_2(x)) = \alpha(\phi_1(x)) + \alpha(\phi_2(x))$$

and thus

$$\alpha_*(\phi_1 + \phi_2)(x) = \alpha_*(\phi_1)(x) + \alpha_*(\phi_2)(x) = (\alpha_*(\phi_1) + \alpha_*(\phi_2))(x).$$

Moreover, α_* is injective: Since α is injective we have $\alpha_*(\phi)(x) = \alpha(\phi(x)) = 0$ if and only if $\phi(x) = 0$ for all $x \in N$, thus $\phi = 0$. Now we still have to show $\ker(\beta_*) = \mathrm{im}(\alpha_*)$.

'$\supseteq$' For $\phi \in \mathrm{Hom}_R(N, M')$ we have $\beta_*(\alpha \circ \phi) = \beta \circ \alpha \circ \phi = 0 \circ \phi = 0$, i.e. $\alpha \circ \phi = \alpha_*(\phi) \in \ker(\beta_*)$.

'$\subseteq$' Let $\phi : N \longrightarrow M$, $\phi \in \ker(\beta_*)$, i.e. $\beta \circ \phi = 0$. We have to show, that there exists $\phi' \in \mathrm{Hom}_R(N, M')$ such that $\phi = \alpha_*(\phi') = \alpha \circ \phi'$. Let $x \in N$. Then $\phi(x) \in \ker(\beta) = \mathrm{im}(\alpha)$. Then there exists $z \in M'$ such that $\phi(x) = \alpha(z)$ and z is unique, since α is injective. Define $\phi'(x) := z$. Then we have $\alpha \circ \phi' = \phi$. It remains to show that ϕ' is R-linear. We have $\phi'(x_1 + x_2) = z$ and with $\alpha(z) = \phi(x_1 + x_2) = \phi(x_1) + \phi(x_2)$ we again have $\alpha(z) = \phi(z_1) + \phi(z_2)$ for some suitable, but unique $z_1, z_2 \in M'$. Since we have

$$\alpha(z) = \phi(x_1 + x_2) = \phi(x_1) + \phi(x_2) = \alpha(z_1) + \alpha(z_2) = \alpha(z_1 + z_2)$$

and α is injective, we have $z = z_1 + z_2$, thus

$$\phi'(x_1 + x_2) = z = z_1 + z_2 = \phi'(x_1) + \phi'(x_2).$$

Moreover for $a \in R$ we have $\phi'(ax) = w$ with $\alpha(w) = \phi(ax) = a \cdot \phi(x) = a \cdot \alpha(z)$. Thus

$$\alpha\left(\phi'(ax)\right) = \alpha(w) = \phi(ax) = a \cdot \phi(x) = a \cdot \alpha(z) = a \cdot \alpha\left(\phi'(x)\right) \overset{\alpha \text{ inj.}}{\Longrightarrow} \phi'(ax) = a \cdot \phi'(x),$$

which proves the claim. □

Remark 11.3 *(i) An R-module N is projective if and only if for every surjective R-linear map $\beta : M \longrightarrow M''$ and every R-linear map $\phi : N \longrightarrow M''$ there is an R-linear map*

$\tilde{\phi} : N \longrightarrow M$, such that the diagram below commutes, i.e. $\phi = \beta \circ \tilde{\phi}$.

$$\begin{tikzcd} & M \arrow[d, "\beta"] \\ N \arrow[ur, dashed, "\tilde{\phi}"] \arrow[r, "\phi"'] & M'' \end{tikzcd}$$

(ii) Free modules are projective.

Definition 11.4 Let M, M_1, M_2 be R-modules. A map

$$\Phi : M_1 \times M_2 \longrightarrow M$$

is called *bilinear*, if the maps

$$\Phi_{x_0} : M_2 \longrightarrow M, \quad y \mapsto \Phi(x_0, y), \qquad \Phi_{y_0} : M_1 \longrightarrow M, \quad x \mapsto \Phi(x, y_0)$$

are linear for all $x_0 \in M_1$ and $y_0 \in M_2$.

Definition 11.5 Let M_1, M_2 be R-modules. A *tensor prodcut* of M_1 and M_2 is an R-module T together with a bilinear map

$$\tau : M_1 \times M_2 \longrightarrow T,$$

such that for every bilinear map $\Phi : M_1 \times M_2 \longrightarrow M$ for any R-module M there is a unique linear map $\phi : T \longrightarrow M$, such that the following diagram becomes commutative.

$$\begin{tikzcd} M_1 \times M_2 \arrow[rr, "\tau"] \arrow[dr, "\Phi"'] & & T \arrow[dl, dashed, "\phi"] \\ & M & \end{tikzcd}$$

Remark 11.6 *Let (T, τ) and (T', τ') be tensor products of R-modules M_1 and M_2. Then there exists a unique isomorphism $h : T \longrightarrow T'$, such that*

$$\tau' = h \circ \tau.$$

proof. Consider

$$\begin{tikzcd} M_1 \times M_2 \arrow[rr, "\tau"] \arrow[dr, "\tau'"'] & & T \arrow[dl, shift left, "h"] \\ & T' \arrow[ur, shift left, "g"] & \end{tikzcd}$$

Existence and uniqueness of the linear maps g and h come from Definition 11.5. It remains to show, that $h \circ g = \mathrm{id}_{T'}$ and $g \circ h = \mathrm{id}_T$.

In order to do this, consider the following diagramm.

$$\begin{tikzcd}
M_1 \times M_2 \arrow[rr, "\tau"] \arrow[dr, "\tau'"'] & & T \arrow[dl, dashed, "g \circ h \overset{!}{=} \mathrm{id}_T"] \\
& T &
\end{tikzcd}$$

We have $(g \circ h)\tau = g \circ (h \circ \tau) = g \circ \tau' = \tau$. By the uniqueness we get $\mathrm{id}_T = g \circ h$. Analogously we get $\mathrm{id}_{T'} = h \circ g$ which finishes the proof. $\square$

Corollary 11.7 *The tensor product (T, τ) of R-modules M_1, M_2 is unique up to isomorphism. The standard notation is*

$$T = M_1 \otimes_R M_2, \qquad \tau(x, y) = x \otimes y$$

Example 11.8 Let M_1, M_2 be free R-modules with bases $\{e_i\}_{i \in I}, \{f_j\}_{j \in J}$. Let T be the free R-module with basis $\{g_{ij}\}_{(i,j) \in I \times J}$ and

$$\tau : M_1 \times M_2 \longrightarrow T, \ (e_i, f_j) \mapsto g_{ij} \quad \text{for all } (i, j) \in I \times J,$$

i.e. for elements in M_1, M_2 we have

$$\tau \left(\sum_{i \in I} a_i e_i, \ \sum_{j \in J} b_j f_j \right) = \sum_{(i,j) \in I \times J} a_i b_j g_{ij}$$

Then (T, τ) is the tensor product of M_1, M_2, since: Let $\Phi : M_1 \times M_2 \longrightarrow M$ be bilinear. Define

$$\phi : T \longrightarrow M, \ g_{ij} \mapsto \Phi(e_i, f_j).$$

Obviously ϕ is linear and satisfies $\Phi = \phi \circ \tau$. Now consider a special case and let $|I| = n, |J| = m$. Identify M_1 via $(e_1, \dots e_n)$ with R^n and M_2 via $(f_1, \dots f_m)$ with R^m. Then T is identified with $R^{n \times m}$ via

$$g_{ij} = E_{ij} = \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & & 1 & & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix}$$

where the only nonzero entry is in the i-th row and j-th column. Then $\tau : R^n \times R^m \longrightarrow R^{n \times m}$ is given by

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = \begin{pmatrix} a_1 b_1 & \dots & a_1 b_m \\ \vdots & & \vdots \\ a_n b_1 & \dots & a_n b_m \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \cdot \begin{pmatrix} b_1 & \dots & b_m \end{pmatrix},$$

where the last multiplication is the usual multiplication of matricees.

Theorem 11.9 *For any two R-modules M_1, M_2 there exists a tensor product $(T,\tau) = (M_1 \otimes_R M_2, \otimes)$.*

proof. Let F be the free R-module with basis $M_1 \times M_2$ and Q be the submodule generated by all the elements

$$(x+x',y)-(x,y)-(x',y), \quad (x,y+y')-(x,y)-(x,y'), \quad (ax,y)-a(x,y), \quad (x,ay)-a(x,y)$$

for $a \in R, x, x' \in M_1, y, y' \in M_2$. Define

$$T := F/Q, \qquad \tau : M_1 \times M_2 \longrightarrow T,\ (x,y) \mapsto \overline{(x,y)}.$$

Then by the construction of Q, τ is bilinear. Let now be M a further R-module and $\Phi : M_1 \times M_2 \longrightarrow M$ a bilinear map. Define

$$\tilde{\phi} : F \longrightarrow M, \quad (x,y) \mapsto \Phi(x,y).$$

Clearly $\tilde{\phi}$ is linear. Moreover we have $Q \subseteq \ker(\phi)$, since Φ is bilinear. By the isomorphism theorem, $\tilde{\phi}$ factors to a linear map $\phi : T \longrightarrow M$ satisfying $\phi\left(\overline{(x,y)}\right) = \Phi(x,y)$. The uniqueness of ϕ follows by the fact that T is generated by the $\overline{(x,y)}$ for $x \in M_1, y \in M_2$. $\square$

Example 11.10 We want to find out what is

$$\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z}.$$

Let $\Phi : \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \longrightarrow A$ bilinear for some $\mathbb{Z}$-module A. Then we see

$$\Phi(\bar{1},\bar{1}) = \Phi(\bar{3},\bar{1}) = \Phi\left(3 \cdot (\bar{1},\bar{1})\right) = 3 \cdot \Phi(\bar{1},\bar{1}) = \Phi(\bar{1},\bar{3}) = \Phi(\bar{1},\bar{0}) = 0 \cdot \Phi(\bar{1},\bar{1}) = 0$$

Hence $\Phi = 0$, since $(\bar{1},\bar{1})$ generates $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. Thus $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} = 0$.

Proposition 11.11 *For R-modules M, M_1, M_2, M_3 we have the following properties.*

(i) $M \otimes_R R \cong M$.

(ii) $M_1 \otimes_R M_2 \cong M_2 \otimes_R M_1$.

(iii) $(M_1 \otimes_R M_2) \otimes_R M_3 \cong M_1 \otimes_R (M_2 \otimes_R M_2)$.

proof. (i) Let $\tau : M \times R \longrightarrow M,\ (x,a) \mapsto a \cdot x$. Then τ is bilinear. We now can verify the universal property of the tensor product. Let N be an arbitrary R-module and $\Phi : M \times R \longrightarrow N$ be bilinear a bilinear map. Define

$$\phi : M \longrightarrow N, \quad x \mapsto \Phi(x,1)$$

Then ϕ is R-linear: For $x, y \in M, \alpha \in R$ we have

$$\phi(\alpha \cdot x) = \Phi(\alpha \cdot x, 1) = \alpha \cdot \Phi(x, 1) = \alpha \cdot \phi(x),$$

$$\phi(x + y) = \Phi(x + y, 1) = \Phi(x, 1) + \Phi(y, 1) = \phi(x) + \phi(y)$$

and thus

$$\phi\left(\tau(x, a)\right) = \phi(a \cdot x) = a \cdot \Phi(x, 1) = \Phi(x, a)$$

(ii) The isomorphism

$$M_1 \times M_2 \xrightarrow{\cong} M_2 \times M_1, \quad (x, y) \mapsto (y, x)$$

induces an isomorphism $M_1 \otimes_R M_2 \cong M_2 \otimes_R M_1$.

(iii) For fixed $z \in M_3$ define

$$\Phi_z : M_1 \times M_2 \longrightarrow M_1 \otimes_R \left(M_2 \otimes_R M_3\right), \quad (x, y) \mapsto x \otimes (y \otimes z) = \tau_{1(23)}\left(\tau_{23}(x, y)\right).$$

Then Φ_z is bilinear and induces a linear map

$$\phi_z : M_1 \otimes_R M_2 \longrightarrow M_1 \otimes_R \left(M_2 \otimes_R M_3\right).$$

Define

$$\Psi : \left(M_1 \otimes_R M_2\right) \times M_3 \longrightarrow M_1 \otimes_R \left(M_2 \otimes_R M_3\right), \quad (x \otimes y, z) \mapsto \phi_z(x \otimes y).$$

Ψ is bilinear and induces a linear map

$$\psi : \left(M_1 \otimes_R M_2\right) \otimes_R M_3 \longrightarrow M_1 \otimes_R \left(M_2 \otimes_R M_3\right)$$

Doing this again the other way round we find a linear map

$$\tilde{\psi} : M_1 \otimes_R \left(M_2 \otimes_R M_3\right) \longrightarrow \left(M_1 \otimes_R M_2\right) \otimes_R M_3$$

By the uniqueness we obtain as in Remark 11.6 that $\psi \circ \tilde{\psi} = \tilde{\psi} \circ \psi = \mathrm{id}$, hence the claim follows. □

Definition + remark 11.12 Let $M, M_1, \ldots M_n$ be R-modules.

(i) A map

$$\Phi : M_1 \times \ldots \times M_n = \prod_{i=1}^{n} M_i \longrightarrow M$$

is called *multilinear*, if for any $1 \leqslant i \leqslant n$ and all choices of $x_j \in M_j$ for $j \neq i$ the map

$$\Phi_i : M_i \longrightarrow M, \quad x \mapsto \Phi(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n)$$

is linear.

(ii) The map

$$\tau_{M_1,\dots M_n} : \prod_{i=1}^{n} M_i \longrightarrow \bigotimes_{i=1}^{n} M_i, \qquad (x_1,\dots,x_n) \mapsto x_1 \otimes \dots \otimes x_n$$

is multilinear.

(iii) For every multilinear map

$$\Phi : \prod_{i=1}^{n} M_i \longrightarrow M$$

there exists a unique linear map

$$\phi : \bigotimes_{i=1}^{n} M_i \longrightarrow M$$

such that $\Phi = \phi \circ \tau_{M_1,\dots M_n}$.

Definition 11.13 Let M, N be R-modules, $\Phi : M^n = \prod_{i=1}^{n} M \longrightarrow N$ a multilinear map.

(i) Φ is called *symmetric*, if for any $\sigma \in S_n$ we have

$$\Phi(x_1, \dots x_n) = \Phi(x_{\sigma(1)}, \dots x_{\sigma(n)}).$$

(ii) Φ is called *alternating*, if

$$x_i = x_j \text{ for some } i \neq j \quad \Longrightarrow \quad \Phi(x_1, \dots x_n) = 0.$$

If $\mathrm{char}(R) \neq 2$, this is equivalent to

$$\Phi(x_1,\dots,x_i,\dots,x_j,\dots,x_n) = -\Phi(x_1,\dots,x_j,\dots,x_i,\dots,x_n).$$

Proposition 11.14 *Let M be an R-module, $n \geqslant 1$.*

(i) There exists an R-module $S^n(M)$, called the n-th symmetric power of M and a symmetric multilinear map

$$\sigma_M^n : M^n \longrightarrow S^n(M)$$

such that for all symmetric, multilinear maps $\Phi : M^n \longrightarrow N$ for any R-module N there exists a unique linear map $\phi : S^n(M) \longrightarrow N$ satisfying $\Phi = \phi \circ \sigma_M^n$.

(ii) There exists an R-module $\Lambda^n(M)$, called the n-th exterior power of M and an alternating multilinear map

$$\lambda_M^n : M^n \longrightarrow \Lambda^n(M)$$

such that for all alternating, multilinear maps $\Phi : \Lambda^n(M) \longrightarrow N$ for any R-module N there exists a unique linear map $\phi : \Lambda^n(M) \longrightarrow N$ satisfying $\Phi = \phi \circ \lambda_M^n$.

proof. (i) Let $T^n(M) = M \otimes_R \ldots \otimes_R M$.

Let now $J_n(M)$ be the submodule of $T^n(M)$ generated by all elements

$$(x_1 \otimes \ldots \otimes x_n) - \left(x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)}\right), \quad x_i \in M, \sigma \in S_n$$

Define

$$S^n(M) := T^n(M) \big/ J_n(M), \qquad \sigma_M^n := \mathrm{proj} \circ \tau_{\mathrm{M,\ldots M}}$$

Then σ_M^n is multilinear and symmetric by construction. Given a multilinear and symmetric map $\Phi : M^n \longrightarrow N$, define ϕ as follows: Let $\tilde{\phi} : T^n(M) \longrightarrow N$ be the linear map induced by Φ and observe that $J_n(M) \subseteq \ker(\tilde{\phi})$. Hence $\tilde{\phi}$ factors to a linear map

$$\phi : S^n(M) = S^n(M) \big/ J_n(M) \longrightarrow N$$

satisfying $\phi \circ \sigma_M^n = \Phi$.

(ii) Similarily let $I_n(M)$ be the submodule of $T^n(M)$ generated by all the elements

$$x_1 \otimes \ldots \otimes x_n, \qquad x_i \in M \;\text{ with } x_i = x_j \;\text{ for some } i \neq j$$

Analogously we define

$$\Lambda^n(M) := T^n(M) \big/ I_n(M), \qquad \lambda_M^n := \mathrm{proj} \circ \tau_{\mathrm{M,\ldots,M}}$$

and obtain the required properties. □

Proposition 11.15 *Let M be a free R-module of rank r and $\{e_1, \ldots, e_r\}$ a basis of M. Then $\Lambda^n(M)$ is a free R-module with basis*

$$\mathrm{proj}(\mathrm{e_{i_1}} \otimes \ldots \otimes \mathrm{e_{i_n}}) =: \mathrm{e_{i_1}} \wedge \ldots \wedge \mathrm{e_{i_n}}, \qquad 1 \leqslant \mathrm{i_1} < \ldots < \mathrm{i_n} \leqslant \mathrm{r}$$

In particular, $\Lambda^n(M) = 0$ for $n > r$ and $\mathrm{rank}\,(\Lambda^{\mathrm{r}}(\mathrm{M})) = 1$.

proof. By definition we have $e_{i_1} \wedge \ldots \wedge e_{i_n} = 0$ if $i_k = i_j$ for some $k \neq j$, hence we have $\Lambda^n(M) = 0$ for $n > r$, as at least on of the e_k must appear twice.

generating: Clearly the $e_{i_1} \wedge \ldots \wedge e_{i_n}, i_k \in \{1, \ldots, r\}$ generate $\Lambda^n(M)$. We have to show that we can leave out some of them. Obviously $e_{i_{\sigma(1)}} \wedge \ldots \wedge e_{i_{\sigma(n)}}$ is a multiple by ± 1 of $e_{i_1} \wedge \ldots \wedge e_{i_n}$. Thus the $e_{i_1} \wedge \ldots \wedge e_{i_n}$ with $1 \leqslant i_1 < i_2 < \ldots < i_n \leqslant r$ generate $\Lambda^n(M)$.

linear independence: Assume

$$\sum_{1 \leqslant i_1 < \ldots < i_n \leqslant r} a_{i_1,\ldots,i_n} e_{i_1} \wedge \ldots \wedge e_{i_n} = 0. \qquad (*)$$

For fixed $j := (j_1, \ldots j_n), 1 \leqslant j_1 < \ldots < j_n \leqslant r$ choose $\sigma_j \in S_r$, such that $\sigma_j(k) = j_k$ for

$1 \leqslant k \leqslant n$. Then we obtain

$$e_{i_1} \wedge \ldots \wedge e_{i_n} \wedge e_{\sigma_j(n+1)} \wedge \ldots \wedge e_{\sigma_j(r)} = \begin{cases} \pm\, e_1 \wedge \ldots \wedge e_r, & \text{if } \ i_k = j_k \ \text{ for all } k \\ 0 & \text{otherwise} \end{cases}$$

By $(*)$ we get

$$0 = \left(\sum_{1 \leqslant i_1 < \ldots i_n \leqslant r} a_{i_1,\ldots,i_n} e_{i_1} \wedge \ldots \wedge e_{i_n} \right) \wedge e_{\sigma_j(n+1)} \wedge \ldots \wedge e_{\sigma_j(r)} = a_j e_{j_1} \wedge \ldots \wedge e_{j_r}$$

and thus $a_j = 0$. □

Example 11.16 Let $M = R^n$. Then $\Lambda^k(M)$ is the free R-module with basis

$$e_{i_1} \wedge \ldots \wedge e_{i_k}, \quad 1 \leqslant i_1 < \ldots < i_k \leqslant n$$

and we have $e_1 \wedge e_2 = -e_2 \wedge e_1$. What is $\Lambda^n(R^n) = \Lambda^n(M)$? And what is λ_k^M? First we obtain $\Lambda^n(R^n) = (e_1 \wedge \ldots \wedge e_n)R \cong R$. Then

$$M^n = (R^n)^n = R^{n \times n}, \quad (a_1, \ldots a_n) = A \in R^{n \times n}, \quad a_i = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{pmatrix} = \sum_{j=1}^n a_{ji} e_j \in R^n = M.$$

For λ_n^M we get

$$\begin{aligned} \lambda_n^M = \lambda_n^{R^n} = \lambda_n(A) &= \lambda_n \left(\sum_{j=1}^n a_{j1} e_j, \ldots, \sum_{j=1}^n a_{jn} e_j \right) \\ &= \sum_{j=1}^n a_{j1} e_j \wedge \ldots \wedge \sum_{j=1}^n a_{jn} e_j \\ &= \sum_{j=1}^n a_{j1} \left(e_1 \wedge \sum_{j=1}^n a_{j2} e_j \wedge \ldots \wedge \sum_{j=1}^n a_{jn} e_j \right) \\ &= \sum_{j=1}^n a_{j1} \cdots \sum_{j=1}^n a_{jn} \left(e_1 \wedge \ldots \wedge e_n \right) \\ &= \sum_{\sigma \in S_n} a_{\sigma(1)1} \cdots a_{\sigma(n)n} \cdot e_1 \wedge \ldots \wedge e_n \cdot \operatorname{sgn}(\sigma) \\ &= \det(A) \cdot e_1 \wedge \ldots \wedge e_n, \end{aligned}$$

which is well-known tu us.

Definition 11.17 Let M be a R-module. Then we define

$$T(M) := \bigoplus_{n=0}^{\infty} T^n(M), \qquad T^0(M) := R, \ T(M) := M$$

$$S(M) := \bigoplus_{n=0}^{\infty} S^n(M). \qquad S^0(M) := R,\ S(M) := M$$

$$\Lambda(M) := \bigoplus_{n=0}^{\infty} \Lambda^n(M), \qquad \Lambda^0(M); = R,\ \Lambda(M) := M$$

On $T(M)$ define a multiplication

$$\begin{array}{rcl} \cdot : T^n(M) \times T^m(M) & \longrightarrow & T^{n+m}(M), \\ (x_1 \otimes \ldots \otimes x_n) \cdot (y_1 \otimes \ldots \otimes y_m) & \mapsto & x_1 \otimes \ldots \otimes x_n \otimes y_1 \otimes \ldots \otimes y_m \end{array}$$

Similarly do it for $S(M)$ and $\Lambda(M)$. Then we have R-algebra-structures and feel free to define

(i) the *tensor algebra* $T(M)$,

(ii) the *symmetric algebra* $S(M)$

(iii) the *exterior algebra* $\Lambda(M)$.

Definition 11.18 Let R be an arbitrary ring.

(i) An R-algebra is a ring R' together with a ring homomorphism $\alpha : R \longrightarrow R'$. In particular R' is an R-module. If α is injective, R'/R is called a *ring extension.*

(ii) A homomorphism of R-algebras R', R'' is an R-linear map $\phi : R' \longrightarrow R''$, which is a ring homomorphism.

Example 11.19 (i) $R[X_1, \ldots X_N]$ is an R-algebra for every $n \in \mathbb{N}$.

(ii) If R' is an R-algebra and $I \trianglelefteq R'$ an ideal, then ${R'}/{I}$ is an R-algebra.

Remark 11.20 *Let R' be an R-algebra, F a free R-module. Then $F' := F \otimes_R R'$ is a free R'-module.*

proof. Let $\{e_i\}_{i \in I}$ be basis of F. Let us show, that $\{e_1 \otimes 1\}_{i \in I}$ is basis of F' as an R-module, where F' is an R' module by

$$b \cdot (x \otimes a) := x \otimes b \cdot a, \qquad a, b \in R,\ x \in F$$

Check the universal property of the free R'-module with basis $\{e_i \otimes 1\}_{i \in I}$ for $F \otimes_R R'$. Let M' be an R-module and $f : \{e_i \otimes 1\}_{i \in I} \longrightarrow M'$ be a map. We have to show: There exists an R'-linear map $\phi : F' \longrightarrow M'$ with $\phi(e_i \otimes 1) = f(e_i \otimes 1)$. Note that the $\{e_i \otimes 1\}$ generate F' as an R'-module, since $e_i \otimes a = a \cdot (e_i \otimes a)$ for $a \in R'$. Let $\tilde{\phi} : F \longrightarrow M'$ be the unique R-linear map satisfying $\tilde{\phi}(e_i) = f(e_i \otimes 1)$. Then define

$$\phi : F \otimes_R R' \longrightarrow M', \quad x \otimes a \mapsto a \cdot \tilde{\phi}(x).$$

Then ϕ is R'-linear an we have

$$\phi(e_i \otimes 1) = 1 \cdot \tilde{\phi}(e_i) = \tilde{\phi}(e_i) = f(e_i \otimes 1),$$

which gives us the desired structure of an R'-module. $\square$

Proposition 11.21 *Let R be a ring, R', R'' two R-algebras.*

(i) $R' \otimes_R R''$ is an R-algebra with multiplication

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) := (a_1 a_2) \otimes (b_1 b_2)$$

(ii) There are R-algebra homomorphisms

$$\sigma' : R' \longrightarrow R' \otimes_R R'', \qquad a \mapsto a \otimes 1$$

$$\sigma'' : R'' \longrightarrow R'' \otimes_R R'', \qquad b \mapsto 1 \otimes b$$

(iii) For any R-algebra A and R-algebra homomorphisms $\phi' : R' \longrightarrow A, \phi'' : R'' \longrightarrow A$, there is a unique R-algebra homomorphism

$$\phi : R' \otimes_R R'' \longrightarrow A$$

satisfying $\phi' = \phi \circ \sigma'$ and $\phi'' = \phi \circ \sigma''$, i.e. making the following diagram commutative

$$\begin{tikzcd}
 & & R' \otimes_R R'' \arrow[dd, dashed, "\phi"] \\
R' \arrow[urr, "\sigma'"] \arrow[drr, "\phi'"'] & R'' \arrow[ur, "\sigma''"] \arrow[dr, "\phi''"] & \\
 & & A
\end{tikzcd}$$

proof. Defining

$$\tilde{\phi} : R' \times R'' \longrightarrow A, \qquad (x, y) \mapsto \phi'(x) \cdot \phi''(y)$$

gives us ϕ, which satisfies the required properties. $\square$

§ 12 Hilbert's basis theorem

Definition 12.1 Let R be a ring, M and R-module.

(i) M is called *noetherian*, if any ascending chain of submodules $M_0 \subset M_1 \subset \ldots$ becomes stationary.

(ii) R is called *noetherian*, if R is noetherian as an R-module, i.e. if every ascending chain of ideals becomes stationary.

Example 12.2 (i) Let k be a field. A k-vector space is noetherian if and only if $\dim(V) < \infty$.

(ii) $\mathbb{Z}$ is noetherian.

(iii) Principle ideal domains are noetherian.

Proposition 12.3 *Let*

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

be a short exact sequence. Then M is noetherian if and only if M' and M'' are noetherian.

proof. '$\Rightarrow$' Let M be noetherian. Let first $M'_0 \subset M'_1 \subset \ldots$ be an ascending chain of submodules in M'. Then $\alpha(M'_0) \subset \alpha(M'_1) \subset \ldots$ is an ascending chain in M. Since M is noetherian, there exists some $n \in \mathbb{N}$, such that $\alpha(M'_i) = \alpha(M'_n)$ for all $i \geqslant n$. Since α is injective, we have $M'_i = M'_n$ for $i \geqslant n$, hence M' is noetherian. Let now $M''_0 \subset M''_1 \subset \ldots$ be an ascending chain of submodules in M''. Then $\beta^{-1}(M_0)'' \subset \beta^{-1}(M''_1) \subset \ldots$ is an ascending chain in M, hence becomes stationary. Since β is surjective, $\beta\left(\beta^{-1}(M''_i)\right) = M''_i$ and thus $M''_0 \subset M''_1 \subseteq \ldots$ becomes stationary.

'$\Leftarrow$' Let $M_0 \subset M_1 \subset \ldots$ be an ascending chain in M. Let $M'_i := \alpha^{-1}(M_i) \cong M_i \cap M'$ and $M''_i := \beta(M_i)$. By assumption, there exists $n \in \mathbb{N}$, such that $M'_i = M'_n$ and $M''_i = M''_n$ for all $i \geqslant n$. Then for $i \geqslant n$ we have

$$\begin{tikzcd}
0 \arrow[r] & M'_n \arrow[r, "\alpha"] \arrow[d, equal] & M_n \arrow[r, "\beta"] \arrow[d, "\gamma"] & M''_n \arrow[r] \arrow[d, equal] & 0 & \text{exact} \\
0 \arrow[r] & M'_i \arrow[r, "\alpha"'] & M_i \arrow[r, "\beta"'] & M''_i \arrow[r] & 0 & \text{exact}
\end{tikzcd}$$

Where γ is injective as an embedding. It remains to show that γ is surjective. Let $z \in M_i$. Since β is surjective, there exists $x \in M_n$, such that $\beta(x) = \beta(z)$. Then $\beta\left(\gamma(x) - z\right) = 0 \Rightarrow \gamma(x) - z = \alpha(y)$ for some $y \in M'_i = M'_n$. Let $\tilde{x} := x - \alpha(y)$. Then

$$\gamma(\tilde{x}) = \gamma(x) - \gamma\left(\alpha(y)\right) = \gamma(x) - \gamma(x) + z = z$$

hence γ is surjective, thus bijective and we have $M_i = M_n$ for $i \geqslant n$. $\square$

Corollary 12.4 *Let R be a noetherian ring.*

(i) Any free R-module F of finite rank n is noetherian.

(ii) Any finitely generated R-module M is noetherian.

proof. (i) Prove this by induction on n.

$n = 1$ Clear.

$n > 1$ Let $e_1, \dots e_n$ be a basis of F and le F' be the submodule generated by $e_1, \dots e_{n-1}$. Then F' is free of rank $n-1$, thus noetherian by induction hypothesis. Moreover ${}^{F}/_{F'}$ is free with generator e_n. Thus we have a short exact sequence

$$0 \longrightarrow F' \longrightarrow F \longrightarrow {}^{F}/_{F'} \longrightarrow 0$$

with $F', {}^{F}/_{F'}$ noetherian, hence by 12.2, F is noetherian.

(ii) If M is generated by $x_1, \dots x_n$, there is a surjective, R-linear map $\phi : F \longrightarrow M$, sending the e_i to x_i, where F is the free R-module with basis $e_1, \dots e_n$. Again by 12.2, M is noetherian which finishes the proof. □

Proposition 12.5 *For an R-module M the following statements are equivalent:*

(i) M is noetherian.

(ii) Any nonempty family of submodules of M has a maximal element with respect to '$\subseteq$'.

(iii) Every submodule of M is finitely generated.

proof. '(i)⇒(ii)' Let $\mathcal{M} \neq \varnothing$ be a set of submodules of M. Let $M_0 \in \mathcal{M}$. If M_0 is not maximal, there is $M_1 \in \mathcal{M}$ with $M_0 \subsetneq M_1$. If M_1 is not maximal, there is $M_2 \in \mathcal{M}$ with $M_1 \subsetneq M_2$. Since M is noetherian, we come to a maximal submodule M_n after finitely many step.

'(ii)⇒(iii)' Let $N \subseteq M$ be a submodule. Let $\mathcal{M}$ be the set of finitely generated submodules of N. Since $(0) \in \mathcal{M}$, we have $\mathcal{M} \neq \varnothing$ and thus there exists a maximal element $N_0 \in \mathcal{M}$. If $N_0 \neq N$, let $x \in N \backslash N_0$ and $N' := N_0 + (x)$ be the submodule generated by N_0 and x. Then clearly $N' \in \mathcal{M}$, which is a contradiction to the maximality of N_0. Hence $N_0 = N$ and N is finitely generated.

'(iii)⇒(i)' Let $M_0 \subseteq M_1 \subseteq \dots$ be an ascending chain of submodules in M. Let $N := \bigcup_{n \in \mathbb{N}_0} M_n$. By assumption, N is finiteley generated, say by $x_1, \dots x_n$. Then there exists $i_0 \in \mathbb{N}$, such that $x_k \in M_{i_0}$ for all $1 \leqslant k \leqslant n$. Thus we have $M_i = M_{i_0}$ for $i \geqslant i_0$, i.e. th chain becomes stationary and M is noetherian. □

Corollary 12.6 *R is noetherian if and only if every ideal $I \trianglelefteq R$ can be generated by finitely many elements. In particular, every principle ideal domain is noetherian.*

proof. Follows from Proposition 12.4. □

Theorem 12.7 (Hilbert's basis theorem) *If R is noetherian, $R[X]$ is also noetherian.*

proof. Let $J \trianglelefteq R[X]$ be an ideal. Assume that J is not finitely generated. Let f_1 be an element of $J\backslash\{0\}$ of minimal degree. Then $(f_1) \neq J$. Inductively let $J_i := (f_1, \dots f_i)$ and pick $f_{i+1} \in J\backslash J_i$ of minimal degree. Let a_i be the leading coefficient of f_i, i.e. we have

$$f_i = a_i X^{\deg(f_i)} + \sum_{j=1}^{\deg(f_i)-1} b_j X^j$$

The ideal $I \trianglelefteq R$ generated by the a_i for $i \in \mathbb{N}$, is finitely generated by assumption. Then we find $n \in \mathbb{N}$ such that $a_{n+1} \in (a_1, \dots, a_n)$, i.e. we have

$$a_{n+1} = \sum_{i=1}^{n} \lambda_i a_i$$

for suitable $\lambda_i \in R$. Let $d_i := \deg(f_i)$. Note, that $d_{i+1} \geqslant d_i$ for all $1 \leqslant i \leqslant n$. Let now

$$\rho := \sum_{i=1}^{n} \lambda_i f_i X^{d_{n+1}-d_i}.$$

Then the leading coefficient of ρ is

$$a_{d_{n+1}} = \sum_{i=1}^{n} \lambda_i a_i$$

Hence $\deg(\rho - f_{n+1}) < d_{n+1}, \rho - f_{n+1} \notin J_n$, since $\rho \in J_n$, so f_{n+1} would be in J_n. This contradicts the choice of f_{n+1}. Hence our assumption was false and J is finitely generated and by Corollary 12.5 $R[X]$ is noetherian.

Corollary 12.8 *Let R be noetherian. Then*

(i) $R[X_1, \dots X_n]$ is noetherian for any $n \in \mathbb{N}$.

(ii) Any finitely generated R-algebra is noetherian.

§ 13 Integral ring extensions

Definition 13.1 Let R be ring, S an R-algebra.

(i) If $R \subseteq S$, S/R is called a *ring extension.*

(ii) If $R \subseteq S$, $b \in S$ is called *integral over* S, if there exists a monic polynomial $f \in R[X] \backslash \{0\}$ such that $f(b) = 0$.

(iii) S/R is called an *integral ring extension*, if every $b \in S$ is integral over R.

Example 13.2 (i) If $R = k$ is a field, then *integral* is equivalent to *algebraic.*

(ii) $\sqrt{2}$ is integral over $\mathbb{Z}$, since $f = X^2 - 2$ is monic with $f(\sqrt{2}) = 0$.

(iii) $\frac{1}{2}$ is not integral over $\mathbb{Z}$.

Assume $\frac{1}{2}$ is integral over $\mathbb{Z}$. Then there exists some monic $f \in R[X]$, such that $f\left(\frac{1}{2}\right) = 0$, i.e. we have

$$\left(\frac{1}{2}\right)^n + g\left(\frac{1}{2}\right) = 0 \ (*)$$

for some $g \in \mathbb{Z}[X]$.Then $2^{n-1} \cdot g\left(\frac{1}{2}\right) \in \mathbb{Z}$. Multiplying $(*)$ by 2^{n-1} gives us

$$2^{n-1} \cdot \left(\left(\frac{1}{2}\right)^n + g\left(\frac{1}{2}\right)\right) = 0$$

and hence

$$\frac{1}{2} = -2^{n-1} \cdot g\left(\frac{1}{2}\right) \in \mathbb{Z}.$$

Thus $\frac{1}{2}$ is not integral over $\mathbb{Z}$. More generally, we easily see that any $q \in \mathbb{Q}\backslash\mathbb{Z}$ is not integral over $\mathbb{Z}$.

Lemma 13.3 *Let S/R be a ring extension, $b \in S$. If $R[b]$ is contained in a subring $S' \subseteq S$ which is finitely generated as an R-module, then b is integral over R.*

proof. Let $s_1, \ldots, s_n$ be generators of S'. Since $b \cdot s_i \in S$ (we have $b \in R[b] \subseteq S$), we find $a_{ik} \in R$, such that

$$b \cdot s_i = \sum_{k=1}^{n} a_{ik} s_k \iff 0 = \sum_{k=1}^{n} (a_i k - \delta_{ik}) s_k. \quad (*)$$

Claim (a) Let A be the coefficient matrix of $(*)$. Then $\det(A) = 0$

Since the determinant is a monic polynomial in b of degree n with coefficients in R, b is integral over R. It remains to show the claim.

(a) Let $A^{\#}$ be the adjoint matrix

$$A^{\#}_{ji} = \det(A_{ij} \cdot (-1)^{i+j}$$

where A_{ij} is obtained from A by deleting the i-the row and j-th column. Recall

$$A^{\#} A = \det(A) \cdot E_n.$$

By $(*)$ we have

$$A \cdot \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix} = 0,$$

hence we have

$$A^{\#} \cdot A \cdot \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix} = 0 \implies \det(A) \cdot s_i = 0 \quad \text{ for all } 1 \leqslant i \leqslant n.$$

Since S' is a subring of S, we have $1 \in S'$, hence there exist $\lambda_1, \ldots, \lambda_n \in R$ with

$$1 = \sum_{i=1}^{n} \lambda_i s_i.$$

Finally

$$\det(A) = \det(A) \cdot 1 = \det(A) \cdot \sum_{i=1}^{n} \lambda_i s_i = \sum_{i=1}^{n} \det(A) \cdot \lambda_i \cdot s_i = 0$$

Proposition 13.4 *Let S/R be a ring extension. Define*

$$\overline{R} := \{b \in S \mid b \text{ is integral over } R\} \supseteq R$$

Then $\overline{R}$ is a subring of S, called the integral closure of R in S.

proof. Let $b_1, b_2 \in \overline{R}$. We have to show, that $b_1 \pm b_2 \in \overline{R}$, $b_1 b_2 \in \overline{R}$. Let $R[b_1]$ be the smallest subring of S containing R and b_1. Then R is finitely generated as an R-module by $1, b_1, b_1^2, \ldots, b_1^{n-1}$, where n denotes the degree of the 'minimal polynomial' of f. Thus $R[b_1, b_2] = (R[b_1])\,[b_2]$ is also finitely generated as an $R[b_1]$-module. This implies, that $R[b_1, b_2]$ is also finitely generated as an R-module and by Lemma 13.2, $R[b_1, b_2]/R$ is an integral ring extension. In particular, $b_1 \pm b_2$ and $b_1 b_2$ are integral over R. □

Definition 13.5 Let S/R be a ring extension, $\overline{R}$ the integral closure of R in S.

(i) R is called *integrally closed* in S, if $\overline{R} = R$.

(ii) Let R be an integral domain. The integral closure of R in $\operatorname{Quot}(R)$ is called the *normalization* of R. R is called *normal*, if it agrees with its normalization.

Proposition 13.6 *Any factorial domain is normal.*

proof. Let R be a domain and $x = \frac{a}{b} \in \operatorname{Quot}(R), a, b \in R, b \neq 0$ relatively prime. Suppose, x is integral over R, i.e. there exist $\alpha_0, \ldots, \alpha_{n-1} \in R$, such that

$$x^n + \alpha_{n-1}x^{n-1} + \ldots + \alpha_1 x + \alpha_0 = 0$$

Multiplying by b^n gives us

$$a^n + \alpha_{n-1}a^{n-1}b + \ldots + \alpha_1 a b^{n-1} + \alpha_0 b^n = 0$$

and hence

$$a^n = b \cdot \underbrace{\left(-\alpha_{n-1}a^{n-1} - \ldots - \alpha_1 a b^{n-2} - \alpha_0 b^{n-1}\right)}_{\in R} \quad \Longleftrightarrow \quad b \mid a^n$$

Since a and b are coprime, we have $b \in R^\times$. Thus $x = \frac{a}{b} = ab^{-1} \in R$ and R is normal. □

Definition 13.7 Let R be a ring.

(i) For a prime ideal $\mathfrak{p} \trianglelefteq R$ we define

$$ht(\mathfrak{p}) := \sup\{n \in \mathbb{N}_0 \;\big|\; \text{there exist prime ideals } \mathfrak{p}_0, \mathfrak{p}_1, \ldots, \mathfrak{p}_n, \text{ with } \mathfrak{p}_n = \mathfrak{p} \text{ and } \mathfrak{p}_0 \subsetneq \ldots \subsetneq \mathfrak{p}_n\}$$

to be the *height* of $\mathfrak{p}$.

(ii) The *Krull-dimension* of R is

$$\dim(R) := \dim_{\text{Krull}}(R) = \sup\{ht(\mathfrak{p}) \;\big|\; \mathfrak{p} \trianglelefteq R \text{ prime }\}$$

Example 13.8 (i) Since $(0) \subsetneq (X_1) \subsetneq (X_1, X_2) \subsetneq \ldots \subsetneq (X_1, \ldots, X_n)$, we have $\dim(k[X_1, \ldots, X_n]) \geqslant n$.

(ii) $\dim(k) = 0$ for any field k, since (0) is the only prime ideal.

(iii) $\dim(\mathbb{Z}) = 1$, since $(0) \subsetneq (p)$ is a maximal chain of prime ideals for $p \in \mathbb{P}$.

(iv) $\dim(R) = 1$ for any principle ideal domain which is not a field:
Assume p, q are prime element with $(p) \subseteq (q)$. Then $p = q \cdot a$ for some $a \in R$. Since p is irreducible, we have $a \in R^\times$ and hence $(p) = (q)$.

(v) $\dim(k[X]) = 1$ for any field k:

Theorem 13.9 (Going up theorem) *Let S/R be an integral ring extension and*

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_n$$

a chain of prime ideals in R. Then there exists a chain of prime ideals

$$\mathfrak{P}_0 \subsetneq \mathfrak{P}_1 \subsetneq \ldots \subsetneq \mathfrak{P}_n$$

in S, such that $\mathfrak{p}_i = \mathfrak{P}_i \cap R$.

proof. Do this by induction on n.

n=0 Let $\mathfrak{p} \lhd R$ be a prime ideal. We have to find a prime ideal $\mathfrak{P} \lhd S$ with $\mathfrak{P} \cap R = \mathfrak{p}$. Let

$$\mathcal{P} := \{I \lhd S \text{ ideal} \,\big|\, I \cap R = \mathfrak{p}\}$$

Claim (a) $\mathfrak{p}S \in \mathcal{P}$.
Then $\mathcal{P}$ is nonempty. Zorn's lemma provides us then a maximal element $\mathfrak{m} \in \mathcal{P}$.
Claim (b) $\mathfrak{m} \lhd S$ is a prime ideal.
This proves the claim. It remains to show the Claims.

(b) Suppose $b_1, b_2 \in S$ with $b_1 b_2 \in \mathfrak{m}$. Assume $b_1, b_2 \in S \backslash \mathfrak{m}$.
Then $\mathfrak{m} + (b_i) \notin \mathcal{P}$, hence $(\mathfrak{m} + (b_i)) \supsetneq \mathfrak{p}$ for $i \in \{1, 2\}$. $\Longrightarrow$ Thus there exists $p_i \in \mathfrak{m}, s_i \in S$ such that $r_i := p_i + b_i s_i \in R \backslash \mathfrak{p}$. Then we have

$$r_1 r_2 = (p_1 + b_1 s_1)(p_2 + b_2 s_2) = \underbrace{p_1 p_2 + p_1 b_2 s_2 + b_1 s_1 p_2}_{\in \mathfrak{m}} + \underbrace{b_1 b_2}_{\in \mathfrak{m} \text{ by ass.}} s_1 s_2 \in \mathfrak{m}$$

Clearly $r_1 r_2 \in R$, hence $r_1 r_2 \in \mathfrak{m} \cap R = \mathfrak{p}$, which is a contradiction, since $\mathfrak{p}$ is prime.

(a) We have to show $\mathfrak{p}S \cap R = \mathfrak{p}$. We prove both inclusions.
'$\supseteq$' This is clear by definition.
'$\subseteq$' Let now

$$b = \sum_{i=0}^{n} p_i t_i, \qquad p_{\in}\mathfrak{p},\ t_i \in S$$

Since the t_i are integral over R, $R[t_1, \ldots t_n] =: S'$ is finitely generated. Let

$s_1, \ldots, s_m$ be generators of S' as an R-module. Since $b \in \mathfrak{p}S'$, we have

$$bs_i = \sum_{k=0}^{m} a_{ki} s_k$$

for suitable $a_{ik} \in \mathfrak{p}$. Then as in lemma 13.3 we have $\det(a_{ik} - \delta_{ik} b) = 0$ and thus b is a zero of monic polynomial with coefficients in $\mathfrak{p}$, i.e. b satisfies an equation

$$b^n + a_{n-1}b^{n-1} + \ldots + a_1 b + a_0 = 0 \qquad \text{with } a_i \in \mathfrak{p},$$

Write

$$b^n = -\sum_{i=0}^{n-1} a_i b^i \in \mathfrak{p},$$

since $b^i \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime, we must have $b \in \mathfrak{p}$ and hence the required inclusion.

n>1 By induction hypothesis we have a chain

$$\mathfrak{P}_0 \subsetneq \mathfrak{P}_1 \subsetneq \ldots \subsetneq \mathfrak{P}_{n-1}$$

satisfying $\mathfrak{P}_i \cap R = \mathfrak{p}_i$. Moreover we find $\mathfrak{P}_n \lhd S$ such that $\mathfrak{P}_n \cap R = \mathfrak{p}_n$. It remains to show $\mathfrak{P}_{n-1} \subsetneq \mathfrak{P}_n$. For $x \in \mathfrak{P}_{n-1}$ we have $x \in R \cap \mathfrak{p}_{n-1}$, i.e. $x \in \mathfrak{p}_{n-1} \subset \mathfrak{p}_n$. Thus $x \in \mathfrak{p}_n \cap R = \mathfrak{P}_n$. Assume now $\mathfrak{P}_{n-1} = \mathfrak{P}_n$. Let $x \in \mathfrak{p}_n$. Then

$$x \in \mathfrak{p}_n \in \mathfrak{p}_n \cap R = \mathfrak{P}_n = \mathfrak{P}_{n-1} = \mathfrak{p}_{n-1} \cap R, \quad \Longrightarrow \quad x \in \mathfrak{p}_{n-1}$$

and thus $\mathfrak{p}_n \subseteq \mathfrak{p}_{n-1}$, hence $\mathfrak{p}_n = \mathfrak{p}_{n-1}$, a contradiction. □

Theorem 13.10 *Let S/R be an integral ring extension. Then* $\dim(R) = \dim(S)$.

proof. '$\leqslant$' Follows from Proposition 13.7

'$\geqslant$' Let $\mathfrak{P}_0 \subsetneq \mathfrak{P}_1 \subsetneq \ldots \subsetneq \mathfrak{P}_n$ be chain of prime ideals in S and define $\mathfrak{p}_i := \mathfrak{P}_i \cap R$.
Then $\mathfrak{p}_i$ is prime and we have $\mathfrak{p}_i \subseteq \mathfrak{p}_{i+1}$. It remains to show, that $\mathfrak{p}_i \neq \mathfrak{p}_{i+1}$.
Define $S' := S/\mathfrak{P}_i$ and $R' := R/\mathfrak{p}_i$. Then S'/R' is integral (!).
We have to show that $\overline{\mathfrak{P}}_{i+1} \cap R = \overline{\mathfrak{p}}_{i+1} :=$ image of $\mathfrak{p}_{i+1}$ in S' is not (0).
Let $b \in \mathfrak{P}_{i+1} \backslash \{0\}$. Since b is integral over R', there exist $a_0, \ldots, a_{n-1} \in R$, such that

$$b^n + a_{n-1}b^{n-1} + \ldots + a_1 b + a_0 = 0$$

Let further n be minimal with this property. Write

$$a_0 = -b \cdot \underbrace{(a_1 + a_2 b + \ldots + a_{n-1}b^{n-2} + b^{n-1})}_{=:c} \in \overline{\mathfrak{P}}_{i+1} \cap R = \overline{\mathfrak{p}}_{i+1}$$

But $c \neq 0$ by the choice of n and $b \neq 0$. Since $R' = R/\mathfrak{p}$ is an integral domain, we have $\overline{0} \neq a_0 \in \overline{\mathfrak{p}}_{i+1}$ and thus $\overline{\mathfrak{p}}_{i+1} \neq (0)$, which proves the claim. $\square$

Theorem 13.11 (Noether normalization) *Let k be a field. Then every finitely generated k-algebra is an integral extension of a polynomial ring over $k[X]$.*

proof. Let $a_1, \dots a_n$ be generators of A as a k-algebra. Prove the theorem by induction.

n=1 If a_1 is transcendental over k, then $A \cong k[X]$. Otherwise $A \cong k[X]/(f)$, where f denotes the minimal polynomial of a_1 over k. Thus A is integral over k.

n>1 If $a_1, \dots a_n$ are algebraically independent, $A \cong k[X_1, \dots X_n]$. Otherwise there exists some polynomial
$F \in k[X_1, \dots X_n] \backslash \{0\}$ such that $F(a_1, \dots a_n) = 0$.

case 1 Assume we have

$$F = X_n^m + \sum_{i=1}^{m-1} g_i X_n^i$$

with $g_i \in k[X_1, \dots X_n]$. Then $F(a_1, \dots a_n) = 0$, hence a_n is integral over $A' := k[a_1, \dots, a_{n-1}]$. By induction hypothesis, A' is integral over some polynomial ring, so is A.

case 2 For the general case write

$$F = \sum_{i=0}^{m} F_i,$$

where F_i is homogenous of degree i, i.e. the sum of the exponents of any monomial in f_i is equal to i. Then replace a_i by $b_i := a_i - \lambda a_n$ (*) with suitable $\lambda_i \in k$, $1 \leqslant i \leqslant n-1$. Then $A \cong k[b_1, \dots, b_{n-1}, a_n]$. For any monomial $a_1^{d_1} \cdots a_n^{d_n}$ we find

$$a_1^{d_1} \cdots a_n^{d_n} = (b_1 + \lambda_1 a_n)^{d_1} \cdots (b_{n-1} + \lambda_{n-1} a_n)^{d_{n-1}} \cdot a_n^{d_n} = \left(\prod_{i=1}^{n-1} \lambda_i^{d_i} \right) \cdot a_n^{\sum_{i=1}^n d_i} + \mathcal{O}(a_n)$$

where $\mathcal{O}(a_n)$ denotes terms of lower degree in a_n. Then for $d := \sum_{i=1}^n d_i$ we obtain

$$F_d(a_1, \dots a_n) = a_n^d \cdot F_d(\lambda_1, \dots \lambda_{n-1}, 1) \ + \ \mathcal{O}(a_n)$$

and thus

$$F(a_1, \dots, a_n) = a_n^m F_m(\lambda_1, \dots, \lambda_{n-1}, 1) \ + \ \mathcal{O}(a_n)$$

Choose now $\lambda_1, \dots, \lambda_{n-1} \in k$, such that $F_m(\lambda_1, \dots, \lambda_{n-1}, 1) \neq 0$. If k is infinite, this is always possible. In the finite case, go back to $(*)$ and use $b_i := a_i + a_n^{\mu_i}$ instead and repeat the procedure. Then by the first case and induction hypothesis the claim follows. $\square$

§ 14 Dedekind domains

Definition 14.1 A noetherian integral domain R of dimension 1 is called a *Dedekind domain*, if every nonzero ideal $I \lhd R$ has a unique representation as a product of prime ideals

$$I \;=\; \mathfrak{p}_1 \cdots \mathfrak{p}_r$$

Definition + remark 14.2 Let R be a noetherian integral domain, $k := \operatorname{Quot}(R)$ and $(0) \neq I \subseteq k$ an R-module.

(i) I is called a *fractional ideal*, if there exists $a \in R\backslash\{0\}$, such that $a \cdot I \subseteq R$.

(ii) I is a fractional ideal if and only if I is finitely generated as an R-module.

(iii) For a fractional ideal I let

$$I^{-1} := \{x \in k \big| x \cdot I \subseteq R\}$$

Then I^{-1} is a fractional ideal.

(iv) I is called *invertible*, if $I \cdot I^{-1} = R$, where $I \cdot I^{-1}$ denotes the R-module generated by all products $x \cdot y$ with $x \in I, y \in I^{-1}$.

proof. (ii) '$\Rightarrow$' If $a \cdot I \subseteq R$, then $a \cdot I$ is an ideal in R. since R is noetherian, $a \cdot I$ is finitely generated, say by $x_1, \ldots, x_n$. Then I is generated by $\frac{x_1}{a}, \ldots, \frac{x_n}{a}$.

'$\Leftarrow$' Let $y_1, \ldots, y_m$ be generators of I. Write $y_i = \frac{r_i}{a_i}$ with $r_i, a_i \in R\backslash 0$. Define

$$a := \prod_{i=1}^{n} a_i$$

Then for any generator we have $a \cdot y_i = r \cdot a_1 \cdot \ldots a_{i-1} \cdot a_{i+1} \cdot \ldots \cdot a_m \in R$, hence $a \cdot I \subseteq R$.

Example 14.3 Every principle ideal $I \neq (0)$ is invertible:
Let $I = (a) \trianglelefteq R$. Then $I^{-1} = \frac{1}{a}R$, since we have

$$I \cdot I^{-1} = (a) \cdot \frac{1}{a}R = aR \cdot \frac{1}{a}R = R$$

Proposition 14.4 *Let R be a Dedekind domain. Then every nonzero ideal $I \trianglelefteq R$ is invertible.*

proof. Let $(0) \neq I \lhd R$ be a proper ideal. Then by assumption we can write

$$I = \mathfrak{p}_1 \cdots \mathfrak{p}_r$$

with prime ideal $\mathfrak{p}_i \lhd R$.
If each $\mathfrak{p}_i$ is invertible, then we have

$$I \cdot \mathfrak{p}_r^{-1} \cdots \mathfrak{p}_1^{-1} = R,$$

hence I is invertible. Thus we may assume that $I = \mathfrak{p}$ is prime. Let $a \in \mathfrak{p}\backslash\{0\}$ and write

$$(a) = \mathfrak{p}_1 \cdots \mathfrak{p}_m$$

with prime ideals $\mathfrak{p}_i \lhd R$. Then $(a) \subseteq \mathfrak{p}$, i.e. $\mathfrak{p}_i \subseteq \mathfrak{p}$ for some $1 \leqslant i \leqslant m$, say $i = 1$. Since the ideals were proper and $\dim(R) = 1$, we have $\mathfrak{p}_1 = \mathfrak{p}$ and $\mathfrak{p}^{-1} = \mathfrak{p}_1^{-1} = \frac{1}{a} \cdot \mathfrak{p}_2 \cdots \mathfrak{p}_m$, since $\mathfrak{p}_1\mathfrak{p}_1^{-1} = \frac{1}{a}(a) = (1) = R$. □

Corollary 14.5 *The fractional ideals in a Dedekind domain R form a group.*

proof. Let $(0) \neq I \subseteq k = \text{Quot}(R)$ ba a fractional ideal. Choose $a \in R$ such that $a \cdot I \subseteq R$. By Proposition 14.3, $a \cdot I$ is invertible, i.e. there exists a fractional ideal I', such that

$$(a \cdot I) \cdot I' = R \implies I \cdot (a \cdot I') = R$$

where R is neutral element of the group. □

Proposition 14.6 *Every Dedekind domain R is normal.*

proof. Let $x \in k := \text{Quot(R)}$ be integral over R, i.e. we can write

$$x^n + a_{n-1}X^{n-1} + \ldots a_1 x + a_0 = 0, \qquad a_i \in R$$

By the proof of Proposition 13.3, $R[x]$ is a finitely generated R-module, hence $R[x]$ is a fractional ideal by Remark 14.2. Further by Corollary 14.4 $R[x]$ is invertible, i.e. we can find $I \trianglelefteq k$, such that $I \cdot R[x] = R$.
On the other hand $R[x]$ is a ring, i.e. $R[x] \cdot R[x] = R[x]$. Multiplying the equation by I gives us $x \in R$. In particular we have

$$R = I \cdot R[x] = I \cdot (R[x] \cdot R[x]) = (I \cdot R[x]) \cdot R[x] = R \cdot R[x] = R[x],$$

which implies the claim. □

Proposition 14.7 *Let R be noetherian integral domain of dimension* 1*. Then R is a Dedekind domain if and only if R is normal.*

proof. '$\Rightarrow$' This is Proposition 14.5
'$\Leftarrow$' We claim

claim (a) For every prime ideal $(0) \neq \mathfrak{p} \lhd R$ the localization $R_\mathfrak{p}$ is a discrete valuation ring.

claim (b) Every nonzero ideal in R is invertible.

Then let $(0) \neq I \neq R$ be an ideal in R. Then $I \subseteq \mathfrak{m}_0$ for a maximal ideal $\mathfrak{m}_0 \lhd R$. By claim (b), $\mathfrak{m}_0$ is invertble. Define $I_1 := \mathfrak{m}_0^{-1} \cdot I$. Then $I_1 \subseteq \mathfrak{m}_0^{-1} \cdot \mathfrak{m}_0 = R$ is an ideal. If $I_1 = R$, then

$I = \mathfrak{m}_0$. Otherwise let $\mathfrak{m}_1$ be a maximal ideal containing I_1 and define $I_2 := \mathfrak{m}_1^{-1} \cdot I_1 \trianglelefteq R$. If $I_1 = I$, then $\mathfrak{m}_0^{-1} \cdot I = I \overset{\text{invert.}}{\Longrightarrow} \mathfrak{m}_0^{-1} = R$, which is a contradiciton.
By this way we obtain a chain of ideals

$$I \subsetneq I_1 \subsetneq I_2 \subsetneq \ldots \subsetneq I_n$$

Since R is noetherian, there exists $n \in \mathbb{N}$; such that $I_n = R$. Then

$$R = I_n = \mathfrak{m}_{n-1}^{-1} \cdot I_{n-1} = \mathfrak{m}_{n-1}^{-1} \cdot \mathfrak{m}_{n-1}^{-1} \cdot I_{n-2} = \mathfrak{m}_{n-1}^{-1} \cdots \mathfrak{m}_0^{-1} \cdot I$$

Thus

$$I = \mathfrak{m}_0 \cdot \mathfrak{m}_1 \cdots \mathfrak{m}_{n-2} \cdot \mathfrak{m}_{n-1}$$

with maximal, thus prime ideals $\mathfrak{m}_i$. Hence R is a Dedekind domain.

It remains to show the claims.

(b) Let $(0) \neq I \trianglelefteq R$ be an ideal. We have to show $I \cdot I^{-1} = R$ for $I^{-1} = \{x \in k \mid x \cdot I \subseteq R\}$.

'$\subseteq$' Clear.

'$\supseteq$' Assume $I \cdot I^{-1} \neq R$. Then there exists a maximal ideal $\mathfrak{m} \lhd R$ such that $I \cdot I^{-1} \subseteq \mathfrak{m}$. By claim (a), $R_\mathfrak{m}$ is a principal ideal domain, thus $I \cdot R_\mathfrak{m}$ is generated by one element, say $\frac{a}{s}$ for some $a \in I, s \in R \backslash \mathfrak{m}$. Let now $b_1, \ldots, b_n$ be generators of I as an ideal in R. Then

$$\frac{b_i}{1} = \frac{a}{s} \cdot \frac{r_i}{s_i}, \quad r_i \in R, s_i \in R \backslash \mathfrak{m}, \quad \text{for } 1 \leqslant i \leqslant n$$

Define $t := s \cdot s_1 \cdots s_n \in R \backslash \mathfrak{m}$.
We have $\frac{t}{a} \in I^{-1}$, since

$$\frac{t}{a} \cdot b_i = \frac{t}{a} \cdot \frac{a}{s} \cdot \frac{r_i}{s_i} = r_i \cdot s_1 \cdots s_{i-1} \cdot s_{i+1} \cdots s_n \in R$$

for $1 \leqslant i \leqslant n$. But then

$$t = \frac{t}{a} \cdot a \in I^{-1} \cdot I \subseteq \mathfrak{m} \quad \lightning$$

(a) We will only give a proof sketch. The strategy is as follows:

(i) Ot suffices to show, that $\mathfrak{m} := \mathfrak{p}R_\mathfrak{p}$ is a principal ideal.

(ii) Show that $\mathfrak{m}^n \neq \mathfrak{m}$.

(iii) Show that $\mathfrak{m}$ is invertible.

Then pick $t \in \mathfrak{m}^2 \backslash \mathfrak{m}$ and obtain $t \cdot \mathfrak{m}^{-1} = R_\mathfrak{m}$. This is true, since otherwise, as $\mathfrak{m}$ is the only maximal ideal in $R_\mathfrak{p}$, we would have $t \cdot \mathfrak{m}^{-1} \subseteq \mathfrak{m}$ and thus $t \in \mathfrak{m}^2$, which implies $\mathfrak{m} = \mathfrak{m}^2$. Then we have

$$(t) = t \cdot R = t \cdot (\mathfrak{m} \cdot \mathfrak{m}^{-1}) = R_\mathfrak{p} \cdot \mathfrak{m} = \mathfrak{m},$$

which will gives us the claim. □

Theorem 14.8 *Let R be a Dedekind domain, L/k a finite separable field extension of $k :=$ Quot(R) and S the integral closure of R in L. Then S is a Dedekind domain.*

proof. We will show all the required properties of a Dedekind domain.
integral domain. This is clear.
dimension 1. We know that S/R is integral and Proposition 13.7 gives us $\dim(S) = 1$.
normal. If $x \in L$ is integral over S, x is integral over R, thus $x \in S$.
noetherian. This is the only hard work in the proof. Let $N := [L : k]$. Since L/k is separable, there exists $\alpha \in L$ such that $L = k(\alpha)$. Moreover we have $\left|\mathrm{Hom}_k(L, \overline{k})\right| = n$, say $\mathrm{Hom}_k(L, \overline{k}) = \{\mathrm{id} = \sigma_1, \dots \sigma_\mathrm{n}\}$.
claim (a) α can be chosen in S.
Then let

$$D := \begin{pmatrix} 1 & \alpha & \dots & \alpha^{n-1} \\ 1 & \sigma_2(\alpha) & \dots & \sigma_2(\alpha^{n-1}) \\ \vdots & \vdots & & \vdots \\ 1 & \sigma_n(\alpha) & \dots & \sigma_n(\alpha^{n-1}) \end{pmatrix} = \left(\sigma_i(\alpha^j)\right)_{(i,j)\in\{1,\dots,n\}\times\{0,\dots,n-1\}}$$

and $d := (\det(D))^2$. $d := d_{L/k}(\alpha)$ is called the *discriminant of L/k with respect to α.*
claim (b) We have

(i) $d \neq 0$

(ii) S is contained in the R-module generated by $\frac{1}{d}, \frac{\alpha}{d}, \dots, \frac{\alpha^{n-1}}{d}$.

Then S is submodule of a finitely generated R-module, and since R is noetherian, S is noetherian as an R-module, thus also as an S-module. This proves *noetherian.* Now prove the claims.

(a) Let $\tilde{\alpha} \in L$ ba a primitive element, i.e. $L = k(\tilde{\alpha})$. Let

$$f = X^n - \sum_{i=0}^{n-1} c_i X^i$$

be the minimal polynomial of $\tilde{\alpha}$ over k. Writr $c_i = \frac{a_i}{b_i}$ for suitable $a_i, b_i \in R, b_i \neq 0$. Now define

$$b := \prod_{i=0}^{n-1} b_i, \qquad \alpha := b \cdot \tilde{\alpha}.$$

Since we have

$$\alpha^n = b^n \tilde{\alpha}^n = b^n \cdot \sum_{i=0}^{n-1} c_i \tilde{\alpha}^i = \sum_{i=0}^{n-1} c_i \cdot \frac{\alpha^i}{b^i} b^n$$

we obtain

$$\alpha^n = b^n \cdot \tilde{\alpha}^n = \sum_{i=0}^{n-1} c_i ? \alpha^i, \quad c_i' = c_i \cdot b^{n-i} \in R.$$

Thus α is integral over R, i.e. $\alpha \in S$. We easily see $k(\alpha) = k(\tilde{\alpha})$, hence the claim is proved.

(b) (i) We have

$$d = (\det(D))^2 = \prod_{1 \leqslant i < j \leqslant n} (\sigma_i(\alpha) - \sigma_j(\alpha))^2 \neq 0,$$

since otherwise we would have $\sigma_i(\alpha) = \sigma_j(\alpha)$, i.e.e $\sigma_i = \sigma_j$, which is not possible.

(ii) Let $\beta \in S$. Write

$$\beta = \sum_{i=0}^{n-1} c_{i+1}\alpha^i, \quad c_i \in k.$$

We have to show: $c_i \in \frac{1}{d}R$ for all $1 \leqslant i \leqslant n$. Therefore we need

claim (c) There is a matrix $A \in R^{n \times n}$ and $b \in R^n$, such that

$$A \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = b \qquad \text{and} \quad \det(A) = d.$$

Then by Cramer's rule and Claim (c) we have

$$c_i = \frac{\det(A_i)}{\det(A)} = \frac{\det(A_i)}{d} \in \frac{1}{d} \in R$$

where A_i is obtained by replacing the i-th column of A by b. This proves claim (b).

(c) Recall that

$$tr_{L/k} : L \longrightarrow k, \quad \beta \mapsto \sum_{i=1}^{n} \sigma_i(\beta)$$

is a k-linear map.For β as above we find for $1 \leqslant i \leqslant n$

$$(*)\ tr_{L/k}(\underbrace{\alpha^{i-1}\beta}_{\in S}) = \sum_{j=1}^{n} tr_{L/k}(\alpha^{i-1}\alpha^{j-1}c_j) = \sum_{j=1}^{n} tr_{L/k}(\alpha^{i-1}\alpha^{j-1})c_j \ \in \ k \cap S = R$$

where the last equality holds since R is normal and by Proposition 14.5. Let now

$$A = (a_{ij})_{(i,i) \in \{1,\ldots,n\} \times \{1,\ldots,n\}}, \quad a_{ij} = tr_{L/k}(\alpha^{i-1}, \alpha^{j-1})$$

and

$$b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}, \quad b_i = Tr_{L/k}(\alpha^{i-1}\beta).$$

Then by $(*)$ we have

$$A \cdot \begin{pmatrix} c_1 \\ \cdots \\ c_n \end{pmatrix} = b,$$

i.e. the first part of the claim. Moreover we have $D^T D = (\tilde{a}_{ij})$, where

$$\tilde{a}_{ij} = \sum_{k=1}^{n} \sigma_k(\alpha^{i-1})\sigma_k(\alpha^{j-1}) = \sum_{k=1}^{n} \sigma_k(\alpha^{i-1}\alpha^{j-1}) = tr_{L/k}(\alpha^{i-1}, \alpha^{j-1}) = a_{ij}.$$

Hence $D^T D = A$ and by $\det(D) = \det(D^T)$ we have

$$\det(D)^2 = \det(D \cdot D) = \det(D \cdot D^T) = \det(A) = d.$$

We have now shown that S is an integral domain, of dimension 1, noetherian and normal. By Proposition 14.6 the theorem is proved. □